NON-CROSSING PARTITIONS OF TYPE (e, e, r)

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Abstract. We investigate a new lattice of generalised non-crossing partitions, constructed using the geometry of the complex reflection group $G(e, e, r)$. For the particular case $e = 2$ (resp. $r = 2$), our lattice coincides with the lattice of simple elements for the type D_n (resp. $I_2(e)$) dual braid monoid. Using this lattice, we construct a Garside structure for the braid group $B(e, e, r)$. As a corollary, one may solve the word and conjugacy problems in this group.

Introduction

The object of this article is the study of the braid group associated with the complex reflection group $G(e, e, r)$, via a new presentation by generators and relations. This new presentation was introduced in our earlier unpublished work [BC], where we proved that it defines a *Garside monoid*, which has many consequences: the monoid embeds in the braid group, the braid group admits nice normal forms, from which one may describe solutions to the word and conjugacy problems. However, our initial proofs in [BC] involved computer assisted case-by-case analysis. While completing [BC], we realised that most computations could be avoided by using a suitable notion of *non-crossing partitions*. The non-crossing partitions of type (e, e, r) form a lattice $NCP(e, e, r)$, with remarkable numerical properties, and sharing many features with the lattice of classical non-crossing partitions (which we denote by $NCP(1, 1, r)$, and which corresponds to the symmetric group). The rich combinatorics of this lattice reflect geometric properties of $G(e, e, r)$. We use these combinatorics to obtain precise structural information about the associated braid group $B(e, e, r)$, but they might also be meaningful in other areas of mathematics where non-crossing partitions have recently appeared, such as cluster algebras and free probabilities.

The classification of finite complex reflection groups was obtained fifty years ago by Shephard and Todd, [ST]. The problem essentially amounts to classifying the irreducible groups, whose list consists of:

- an infinite family $G(de, e, r)$, where d, e, r are arbitrary positive integral parameters;
- 34 exceptions, labelled $G_4, \ldots, G_{37}$.

The infinite family includes the four infinite families of finite Coxeter groups: $G(1, 1, r) \simeq W(A_{r-1})$, $G(2, 1, r) \simeq W(B_r)$, $G(2, 2, r) \simeq W(D_r)$ and $G(e, e, 2) \simeq W(I_2(e))$. For all other values of the parameters, $G(de, e, r)$ is an irreducible monomial complex reflection group of rank r, with no real structure.

A general problem is to generalise to complex reflection groups as much as possible from the theory of Weyl groups and Coxeter groups. One of the motivations is the *Spetses* program of Broué-Malle-Michel (see [BMM]), which formally extends certain representation-theoretic aspects of reductive algebraic groups, as if certain complex reflection groups were the "Weyl groups" of more general structures (the *Spetses*). Not all complex reflection groups give rise to Spetses. In

the infinite family, one considers, in addition to the real groups, the complex subfamilies $G(d,1,r)$ and $G(e,e,r)$. Among recent developments is the description by Broué-Kim, [BK], of the "Lusztig families of characters" of $G(d,1,r)$ and $G(e,e,r)$. These representation-theoretic aspects involve generalised braid groups and Hecke algebras, which were introduced by Broué-Malle-Rouquier. Following [BMR], one defines the *braid group* $B(G)$ attached to a complex reflection group G as the fundamental group of the space of regular orbits.

When G is real, the braid group $B(G)$ is well understood thanks to Brieskorn's presentation theorem and the subsequent structural study by Deligne and Brieskorn-Saito ([Br], [D], [BrS]). The braid group of $G(d,1,r)$ is isomorphic to the type B_r Artin group, hence is also well understood. Our object of interest is the braid group of $G(e,e,r)$. Note that this subseries contains the D-type and I_2-type Coxeter series.

Using an induction based on fibration arguments, a presentation for the braid group of $G(e,e,r)$ was computed in [BMR]. This presentation shares with those of finite type Artin groups (= braid groups of finite Coxeter groups) the following features: there are r generators, which correspond to *braid reflections* (also called *generators of the monodromy*), relations are positive and homogeneous, and addition of quadratic relations yields a presentation of the reflection group. But when both $e \geq 3$ and $n \geq 3$ the monoid defined by this presentation fails to embed in the braid group (see [C1], p. 122). By contrast with finite type Artin presentations, Broué-Malle-Rouquier's presentation is not *Garside*, in the sense of Dehornoy-Paris, [DP]. The techniques of Deligne and Brieskorn-Saito are not applicable to this presentation.

In [BKL], [BDM] and [B1], new Garside presentations for finite type Artin group were introduced. Geometrically, the generating sets naturally arise when "looking" at the reflection arrangement from an eigenvector for the Coxeter element. In $G(e,e,r)$, maximal order regular elements (in the sense of [Sp]) can be seen as analogues of Coxeter elements. Our new presentation for $B(e,e,r)$ is analog of the presentation from [B1]. The generators are *local generators* associated with eigenvectors of such generalised Coxeter elements (as in *loc. cit.*, section 4). In particular, when restricted to Coxeter types D_r and $I_2(e)$, our monoid coincides with the dual braid monoid.

The lattice $NCP(e,e,r)$ coincides with the lattice of *simple elements* of our monoid. Its numerology (cardinality, Möbius number, Zeta function) may be described in terms of the reflection degrees of $G(e,e,r)$, following the pattern observed by Chapoton for the Coxeter types ([Ch]). In particular, the cardinality of $NCP(e,e,r)$ is a *generalised Catalan number*. When G is a Weyl group, the numerical invariants of the dual braid monoid are connected with those of cluster algebras (see again [Ch]). This is another evidence suggesting to look for analogs of representation-theoretic objects, where the Weyl group is replaced by a spetsial complex reflection group. Another area of mathematics where non-crossing partitions (associated with $G(1,1,r)$ and $G(2,1,r)$) have recently appeared is the theory of free probabilities ([BGN]) – is it possible to use $NCP(e,e,r)$ to construct a free probability theory?

There are two possible ways of proving that a certain monoid is a Garside monoid. One possibility is to use the notion of *complete presentations* (see [Dh2]); this purely word-theoretical approach, used in our earlier work [BC], is applicable in quite general situations, but does not provide intuitive explanation of why the result is true. The second approach, used for example in [BDM], is to find a good interpretation of the poset of simple elements, which allows a direct proof of the lattice property. This second approach is used here. Our strategy resembles the one used in [BDM] and [B1].

The first section contains the construction of our lattice $NCP(e,e,r)$, and a list of its basic properties; it may be read independently from the rest of the article. Given any finite subset $x \subset \mathbb{C}$, one may define a natural notion of *non-crossing partition of* x. These partitions are naturally ordered by refinement. The non-crossing partitions of type (e,e,r) are particular non-crossing partitions of the set $\mu_{e(r-1)} \cup \{0\}$ (a regular $e(r-1)$-gon, together with its center 0), which are symmetric or asymmetric in a particular way. Even though the lattice $NCP(e,e,r)$ is related to the geometry of $G(e,e,r)$, its description is purely combinatorial. The case $e=2$ gives a notion of non-crossing partitions of type D_r, which does not coincide with initial definition by Reiner ([R]), but is equivalent to the independent new definition by Athanasiadis-Reiner ([AR]) and provides a geometric interpretation for the simple elements of the dual monoid of type D_r. When $r=2$, we also obtain a geometric interpretation for the dual monoid of type $I_2(e)$. As expected, there is an analog of Kreweras complement operation: given $u \preccurlyeq v$ in $NCP(e,e,r)$, we define a new element $u\backslash v$.

In the second section, we construct natural maps from $NCP(e,e,r)$ to $B(e,e,r)$ and $G(e,e,r)$. The construction relies on the choice of a generalised Coxeter element $c \in G(e,e,r)$. The map $u \mapsto b_u$ satisfies the property $b_u b_{u\backslash v} = b_v$ for all $u \preccurlyeq v$ in $NCP(e,e,r)$ (Proposition 2.4). This provides an *a posteriori* justification for using a "left-quotient" notation for the complement operation.

In section 3, we have a closer look at minimal non-trivial partitions and their images in $B(e,e,r)$ and $G(e,e,r)$. This prepares for section 4, which contains a crucial step in our construction. Consider the set T of all reflections in $G(e,e,r)$. There is a notion of *reduced T-decompositions* of an element $g \in G$. Define a relation $\preccurlyeq_T$ on $G(e,e,r)$, by setting $g \preccurlyeq_T h$ whenever g is the product of an initial segment of a reduced T-decomposition of h. We prove that the map from $NCP(e,e,r)$ to $G(e,e,r)$ lands in $P_G := \{g \in G(e,e,r) | g \preccurlyeq c\}$. Moreover, it induces a poset isomorphism $(NCP(e,e,r), \preccurlyeq) \simeq (P_G, \preccurlyeq_T)$. In particular, $(P_G, \preccurlyeq_T)$ is a lattice, generalising what happens in Coxeter groups.

Once this lattice property is proved, the Garside structure for $B(e,e,r)$ is obtained very easily (section 5). Besides the many general properties of Garside structures (solution to the word and conjugacy problems, finite classifying spaces, etc...) which we do not detail, we mention another application: the computation of certain centralisers, following a conjecture stated in [BDM].

Another application is the description of an explicit presentation. In section 7, we give such a presentation; generators correspond to minimal non-discrete partitions, and relations are of length 2. This follows the pattern observed in [B1] for real reflection groups. Here again, the fact that it is enough to consider relations of length 2 is a consequence of the transitivity of the classical braid group Hurwitz action on the set of reduced T-decompositions of c (section 6).

Section 8 deals with enumerative aspects of $NCP(e,e,r)$.

In the final section, we return to the study of Broué-Malle-Rouquier's presentation. Not only does t not satisfy the embedding property, but it is impossible to add a finite number of relations (without changing the generating set) to obtain a presentation with the embedding property.

For real reflection groups, both the classical "Coxeter-Artin" viewpoint and the dual approach provide Garside structures. Here only the dual approach is applicable. The main features of our construction actually generalise to all well-generated complex reflection groups (see [B2], and its sequel in preparation).

Notations and terminology

Throughout this article, e and n are fixed positive integers. Rather than considering $G(e,e,n)$, it simplifies notations to work with $G(e,e,n+1)$.

We denote by $|X|$ the cardinal of a set X.

If d is a positive integer, we denote by μ_d the group of complex d-th roots of unity. We denote by ζ_d the generator $\exp(2i\pi/d)$ of μ_d.

When X is a finite set, a cyclic ordering on X consists of a subset of X^3 (whose elements are called *direct triples*) subject to certain (obvious) axioms. E.g., the set μ_d comes equipped with a standard counterclocwise cyclic ordering. Any cyclic ordering on a finite set X is equivalent, via a bijection $X \simeq \mu_{|X|}$, to this standard example (we may use this property as a definition, and avoid stating explicitly the axioms).

When X is endowed with a cyclic ordering and $x_1, x_2 \in X$, we denote by $\langle x_1, x_2 \rangle$ the set of $x \in X$ such that (x_1, x, x_2) is direct. We use the similar notations $\rangle x_1, x_2 \rangle$, $\langle x_1, x_2 \langle$ and $\rangle x_1, x_2 \langle$ to exclude one or both of the endpoints.

In [BDM] and [B1], the symbol $\prec$ is used to denote various order relations. The symbol $\preccurlyeq$ is used here for the same purposes. When we write $u \prec v$, we mean $u \preccurlyeq v$ and $u \neq v$.

1. Generalised non-crossing partitions

Our goal here is to construct a suitable theory of non-crossing partitions of type $(e,e,n+1)$.

While our construction is motivated by geometric considerations on complex reflection groups, it is possible to give purely combinatorial definitions. Formally, the reader solely interested in combinatorics of $NCP(e,e,n+1)$ can read this section without any knowledge of complex reflection groups.

1.1. **Set-theoretical partitions.** Let x be a set. A *(set-theoretical) partition* of x is an unordered family $u = (a)_{a \in u}$ of disjoint non-empty subsets of x (the "parts") such that $x = \bigcup_{a \in u} a$. Partitions are in natural bijection with abstract equivalence relations on x.

The set P_x of partitions of x is endowed with the following order relation:

Definition 1.1. *Let u, v be partitions of x. We say that "u refines v", and write $u \preccurlyeq v$, if and only if*

$$\forall a \in u, \exists b \in v, a \subseteq b.$$

We recall that a *lattice* is a poset $(P, \leq)$ such that, for any pair of elements $u, v \in P$, there exists elements $u \wedge v$ (read "u meet v") and $u \vee v$ (read "u join v") in P, such that $u \wedge v \leq u, u \wedge v \leq v$, and $\forall w \in P, (w \leq u)$ and $(w \leq v) \Rightarrow w \leq u \wedge v$, and symmetrically for $u \vee v$.

Clearly, $(P_x, \preccurlyeq)$ is a lattice. For any $u, v \in P_x$, $u \wedge v$ is the partition whose parts are the non-empty $a \cap b$, for $a \in u$ and $b \in v$. The element $u \vee v$ is better understood in terms of equivalence relations: it is the smallest equivalence relation whose graph contains the graphs of u and v. The minimal element of P_x is the *discrete* partition "disc" whose parts are singletons. The maximal element of P_x is the *trivial* partition "triv" with the unique part x.

In the sequel, we will consider several subsets of P_x. They will always be endowed with the order relation obtained by restricting $\preccurlyeq$. Regardless of the subset of P_x, we will always denote this restriction by $\preccurlyeq$. Our point will be to prove that the considered subsets are lattices. In all cases, it will be enough to rely on the following easy lemma to prove the lattice property.

Lemma 1.2. *Let $(P, \leq)$ be a finite lattice. Let $Q \subseteq P$, endowed with the restricted order. Denote by m the maximal element of P. Denote by $\wedge_P$ the meet operation in P. Assume that*

$$m \in Q \qquad \text{and} \qquad \forall u, v \in Q, u \wedge_P v \in Q.$$

Then $(Q, \leq)$ is a lattice, with meet operation $\wedge_Q = \wedge_P$, and join operation $\vee_Q$ satisfying

$$\forall u, v \in Q, u \vee_Q v = \bigwedge_{\substack{w \in Q \\ u \leq w, v \leq w}} w.$$

In such a context, it is convenient to use the notation $\wedge$ to refer to both $\wedge_P$ and $\wedge_Q$.

1.2. **Non-crossing partitions of a configuration of points.** Let x be a finite subset of $\mathbb{C}$. A partition λ of x is said to be non-crossing if it satisfies the following condition:

For all $\nu, \nu' \in \lambda$, if the convex hulls of ν and ν' have a non-empty intersection, then $\nu = \nu'$.

We denote by NCP_x the set of non-crossing partitions of x.

Lemma 1.3. *The poset $(NCP_x, \preccurlyeq)$ is a lattice.*

Proof. By Lemma 1.2, it is enough to check that the trivial partition is in NCP_x and that the meet of two non-crossing partitions is non-crossing. Both follow immediately from the definition. □

1.3. **Non-crossing partitions of type $(1, 1, n)$.** The notation $NCP(1, 1, n)$ is just another name for the lattice of non-crossing partitions of a regular n-gon; this very classical object is related to the geometry of the symmetric group $\mathfrak{S}_n$, aka $W(A_{n-1})$ or $G(1, 1, n)$:

Definition 1.4. *For any positive integer n, we set $NCP(1, 1, n) := NCP_{\mu_n}$.*

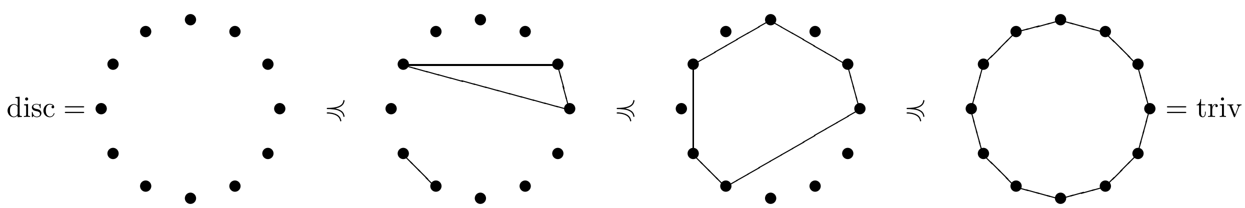

FIGURE 1. Example of a chain in $NCP(1, 1, 15)$

By specialising Lemma 1.3, we obtain:

Lemma 1.5. *The poset $(NCP(1, 1, n), \preccurlyeq)$ is a lattice.*

Our next task is to define the *complement* operation for $NCP(1, 1, n)$. This construction will later be generalised to $NCP(e, e, n)$ (subsection 1.7).

We start with a particular case:

Definition 1.6. *Let $a \subset \mu_n$. Let $x_1, \ldots, x_k$ be the elements of a, ordered counterclockwise. We denote by $\overline{a}$ the element of $NCP(1, 1, n)$ whose parts are*

$$\rangle x_1, x_2\rangle\ ,\ \rangle x_2, x_3\rangle\ ,\ \ldots\ ,\ \rangle x_{r-1}, x_r\rangle\ ,\ \rangle x_r, x_1\rangle.$$

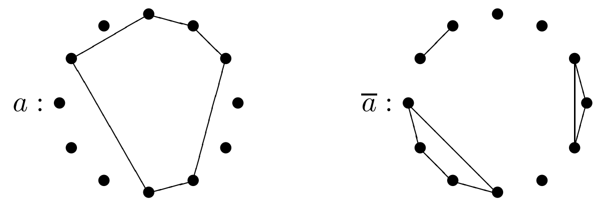

FIGURE 2. The map $a \mapsto \overline{a}$, when $a \subseteq \mu_n$

The general situation is as follows: Let $u, v \in NCP(1,1,n)$, with $u \preccurlyeq v$. Let us recursively define an element $u\backslash v \in NCP(1,1,n)$ by induction on the number of non-singleton parts of u:

(i) If u has only singleton parts, we set $u\backslash v := v$.

(ii) If u has only one non-singleton part a, denote by b the unique part of v such that $a \subseteq b$. Let ϕ a cyclic-order preserving bijection from b to $\mu_{|b|}$. Using Definition 1.6, we obtain an element $\overline{\phi(a)} \in NCP_{\mu_{|b|}}$. Transporting it via ϕ, we obtain a non-crossing partition $\phi^{-1}(\overline{\phi(a)})$ of b. We set $u\backslash v$ to be the partition obtained from v by splitting the part b into $\phi^{-1}(\overline{\phi(a)})$.

(iii) If u has several non-singleton parts, choose a one of them. Let $u' \in NCP(1,1,n)$ be the partition obtained from u by splitting a into isolated points, let $u'' \in NCP(1,1,n)$ with a as only non-singleton part. We set $u\backslash v := u'\backslash(u''\backslash v)$.

For Step (iii) to make sense, one has to observe that $u' \preccurlyeq u''\backslash v$, which is clear (look at figure 2). The procedure is non-deterministic, since one has to make choices when applying Step (iii). However, one may easily check that the final result does not depend on these choices.

An obvious property is that, when $u \preccurlyeq v$, we also have $(u\backslash v) \preccurlyeq v$.

Definition 1.7. *Let $u, v \in NCP(1,1,n)$ such that $u \preccurlyeq v$. The non-crossing partition $u\backslash v$ is called the* complement of u in v.

Generalising Definition 1.6, we set, for all $u \in NCP(1,1,n)$, $\overline{u} := u\backslash$ triv*, where* triv *is the maximal element of $NCP(1,1,n)$.*

An alternative but equivalent definition for the complement is given in [BDM]. The original construction of the complement actually goes back to Kreweras [K].

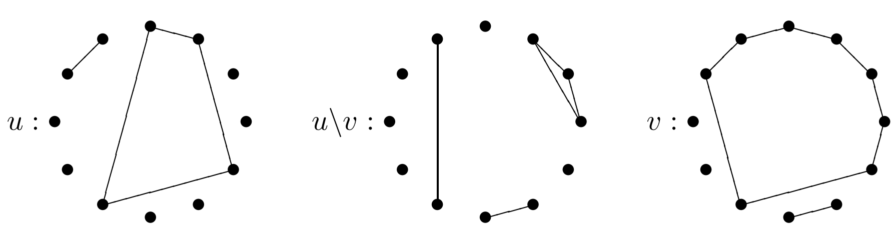

FIGURE 3. The complement: an example

Explanation. Though there is no logical need for this yet, interpreting the "complement" operation in terms of permutations and braids provides useful intuition. To illustrate this, we briefly

recall, without proofs, how this construction is done [BDM] (this anticipates on what will be done in Section 2 for $G(e,e,n+1)$). To any element $u \in NCP(1,1,n)$, we associate the permutation of $\sigma_u \in \mu_n$ sending any $\zeta \in \mu_n$ to ζ', the successor of ζ for the counterclockwise ordering of the part of u containing ζ. If we compose permutations the way we compose paths ($\sigma\tau$ meaning "σ then τ"), we may check that the relation $\sigma_u \sigma_{u\backslash v} = \sigma_v$, for all $u \preccurlyeq v$ in $NCP(1,1,n)$. Since the map $u \mapsto \sigma_u$ is injective, $u\backslash v$ is uniquely determined by the equation $\sigma_u \sigma_{u\backslash v} = \sigma_v$. More remarkably, the analog relation holds in the usual braid group on n strings, when one associates to u the braid b_u represented by motion of the points of μ_n where each ζ goes to the ζ' constructed above, following at constant speed the affine segment $[\zeta, \zeta']$ (with the convention that, if the part of ζ and ζ' contains only two elements, then the strings of ζ and ζ' avoid each other by "driving on the right" along $[\zeta, \zeta'])$. A graphical illustration is provided in Figure 4.

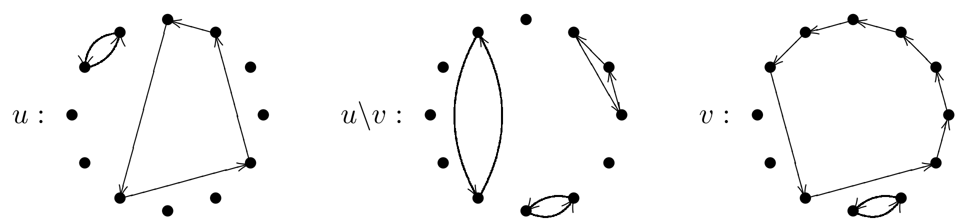

FIGURE 4. The crucial property of the complement: the composition of the motion associated with u with the motion associated with $u\backslash v$ is homotopic to the motion associated with v.

Note that the group μ_n naturally acts on $NCP(1,1,n)$, by its multiplicative action on the underlying set μ_n. The map $u \mapsto \overline{u}$ is a "square root" of the multiplication by ζ_n:

Lemma 1.8. *For all $u \in NCP(1,1,n)$, we have $\overline{\overline{u}} = \zeta_n u$.*

Proof. Though it is possible to prove this in a purely combinatorial way, we give a simple proof using the interpretation in terms of permutations. One has $\sigma_u \sigma_{\overline{u}} = \sigma_{\overline{u}} \sigma_{\overline{\overline{u}}} = \sigma_{\text{triv}}$, thus $\sigma_u \sigma_{\text{triv}} = \sigma_u \sigma_{\overline{u}} \sigma_{\overline{\overline{u}}} = \sigma_{\text{triv}} \sigma_{\overline{\overline{u}}}$ and $\sigma_{\overline{\overline{u}}} = \sigma_{\text{triv}}^{-1} \sigma_u \sigma_{\text{triv}}$. We observe that σ_{triv} is an n-cycle, corresponding to a rotation. Conjugating by this n-cycle amounts to relabelling the underlying set by rotation. $\square$

It is well-known that the cardinality of $NCP(1,1,n)$ is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. We will have a natural interpretation for the cardinality of our generalised non-crossing partitions lattices.

Definition 1.9. *Let $u \in NCP(1,1,n)$. The* height *of u is the integer $ht(u) := n - m$, where m is the number of parts of u.*

The following lemma follows straightforwardly from the definitions:

Lemma 1.10. *For all $u, v \in NCP(1,1,n)$, we have*

- $0 \leq ht(u) \leq n-1$, $ht(\text{disc}) = 0$, $ht(\text{triv}) = n-1$;
- $u \preccurlyeq v \Rightarrow ht(u) \leq ht(v)$ *and* $u \prec v \Rightarrow ht(u) < ht(v)$;
- *if* $u \preccurlyeq v$, *then* $ht(u) + ht(u\backslash v) = ht(v)$; *in particular,* $ht(u) + ht(\overline{u}) = n-1$.

1.4. **Non-crossing partitions of type** $(e, 1, n)$**.** For any divisor e of n, the group μ_e acts on $NCP(1, 1, n)$ by multiplication on the underlying set μ_n. We denote by $NCP(1, 1, n)^{\mu_e}$ the set of non-crossing partitions fixed by the action of μ_e. The following definition was already implicit in [BDM].

Definition 1.11. *For any positive integers* e, n*, we set* $NCP(e, 1, n) := NCP(1, 1, en)^{\mu_e}$*.*

Lemma 1.12. *The poset* $(NCP(e, 1, n), \preccurlyeq)$ *is a lattice.*

The reflection group $G(2, 1, n)$ is also known as the "hyperoctahedral group" $W(B_n)$. When $e = 2$, the above definition coincides with Reiner's type B_n non-crossing partitions [R], and with the lattice of simple elements in the type B_n dual braid monoid [B1].

Let $u, v \in NCP(e, 1, n)$, with $u \preccurlyeq v$. The element $u \backslash v$ (defined earlier in $NCP(1, 1, en)$) is clearly in $NCP(e, 1, n)$.

Definition 1.13. *An element of* $NCP(e, 1, n)$ *is* long *if* 0 *is in the convex hull of one of its parts (by non-crossedness, this part must be unique; it is referred to as* the long part *of the partition). An element of* $NCP(e, 1, n)$ *which is not long is* short*.*

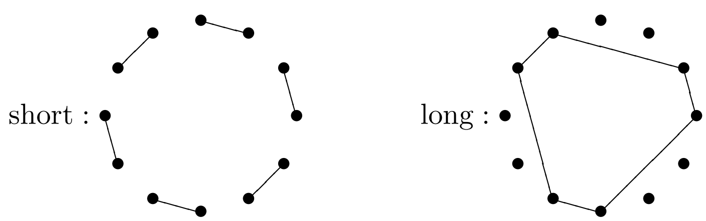

FIGURE 5. Some elements of $NCP(3, 1, 5)$

Lemma 1.14. *The map* $u \mapsto \overline{u}$ *sends long elements of* $NCP(e, 1, n)$ *to short elements, and vice-versa. In particular, the number of long elements equals the number of short elements.*

The complement of a short element in a short element is short, the complement of a long element in a long element is short, the complement of a short element in a long element is long.

Proof. Use the height function $ht : NCP(1, 1, en) \to \mathbb{N}$. We have $ht(u) = en - m_u$, where m_u is the number of parts of $u \in NCP(e, 1, n)$.

If u is short, then $m_u \equiv 0[e]$. If u is long, we have $m_u \equiv 1[e]$ (only the long part is fixed by μ_e-action). We have (Lemma 1.10) $ht(u) + ht(\overline{u}) = en - 1 = 2en - m_u - m_{\overline{u}}$. Among u and $\overline{u}$, one is short and the other is long.

The remaining statements are proved similarly. $\square$

Remark. When studying $NCP(e, 1, n)$, it is natural to work with a new length function ht', defined by $ht'(u) := n - m_u/e$ when u is short and $ht'(u) := n - (m_u - 1)/e$ when u is long. This function takes its values in $\{0, \ldots, n\}$.

1.5. **Non-crossing partitions of type** $(e,e,n+1)$**.** In this subsection, we define non-crossing partitions of type $(e,e,n+1)$ as being non-crossing partitions of $\mu_{en}\cup\{0\}$ satisfying certain additional conditions. The geometry of $NCP(e,e,n+1)$ will be related to that of $NCP(e,1,n)$ by three natural poset morphisms, according to the diagram:

$$\begin{tikzcd} NCP(e,1,n) \arrow[r, hook, "*"] & NCP(e,e,n+1) \arrow[l, two heads, bend right, "\sharp"'] \arrow[l, two heads, bend left, "\flat"] \end{tikzcd}$$

Definition 1.15. *For all* $u\in NCP_{\mu_{en}\cup\{0\}}$*, we denote by* $u^\flat\in NCP_{\mu_{en}}$ *the partition obtained by forgetting* 0*. We set*

$$NCP(e,e,n+1) := \{u\in NCP_{\mu_{en}\cup\{0\}} | u^\flat \in NCP(e,1,n)\}.$$

An element $u\in NCP(e,e,n+1)$ *is said to be:*

- short symmetric*, if* $u^\flat$ *is a short element of* $NCP(e,1,n)$ *and* $\{0\}$ *is a part in* u*;*
- long symmetric*, if* $u^\flat$ *is a long element of* $NCP(e,1,n)$*. The non-crossing condition then implies that* $a\cup\{0\}$ *is a part of* u*, where* a *is the long part of* $u^\flat$*; we say that* $a\cup\{0\}$ *is* the long part *of* u*;*
- asymmetric*, if* $u^\flat$ *is a short element of* $NCP(e,1,n)$*, and* $\{0\}$ *is a not a part of* u*. There is then an unique part* a *of* $u^\flat$ *such that* $a\cup\{0\}$ *is a part of* u*; the part* $a\cup\{0\}$ *is* the asymmetric part *of* u*.*

Clearly, these three cases are mutually exclusive and any element of $NCP(e,e,n+1)$ is of one of the three types.

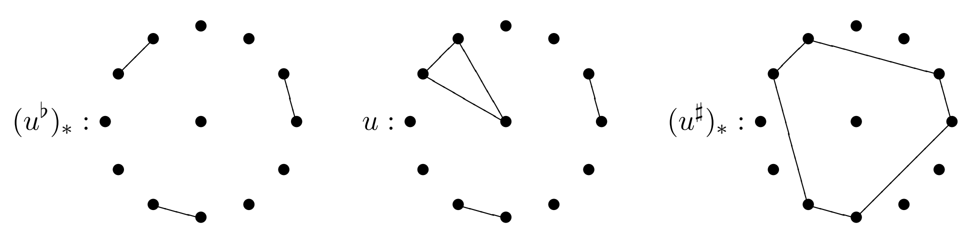

FIGURE 6. Some elements of $NCP(3,3,5)$: short symmetric, asymmetric and long symmetric.

From the definition, it follows immediately that if u is short symmetric, v asymmetric and w long symmetric, then one cannot have $v\preccurlyeq u$ nor $w\preccurlyeq v$ nor $w\preccurlyeq u$.

The subposet of $NCP(e,e,n+1)$ consisting of symmetric elements, both short and long, is isomorphic to $NCP(e,1,n)$ via the map $NCP(e,1,n)\to NCP(e,e,n+1), u\mapsto u_*$, with u_* defined as follows:

- If u is short, we set $u_* := u\cup\{\{0\}\}$. This identifies short elements in $NCP(e,1,n)$ with short symmetric elements in $NCP(e,e,n+1)$.
- If u is long and if a is the long part of u, the partition u_* is obtained from u by replacing a by $a\cup\{0\}$. This identifies long elements in $NCP(e,1,n)$ with long symmetric elements in $NCP(e,e,n+1)$.

Finally, we define a map $NCP(e,e,n+1) \to NCP(e,1,n), \lambda \mapsto \lambda^\sharp$ as follows:

- If u is symmetric, then we set $u^\sharp := u^\flat$.
- If u is asymmetric, let a be the part containing 0. Let $\tilde{a} := (\bigcup_{\zeta\in\mu_e} \zeta a) - \{0\}$. We set $u^\sharp$ to be the element of $NCP(e,1,n)$ containing $\tilde{a}$ as a part, and apart from that made of parts in u.

From the definitions, it immediately follows that, for all $u \in NCP(e,1,n)$ and for all $v \in NCP(e,e,n+1)$:

$$(u_*)^\flat = u \quad \text{and} \quad (u_*)^\sharp = u$$

$$(v^\flat)_* \preccurlyeq v \preccurlyeq (v^\sharp)_*$$

$$(v^\flat)_* = v \Leftrightarrow (v^\sharp)_* = v \Leftrightarrow v \text{ is symmetric.}$$

Not only are the maps $\flat$ and $\sharp$ retractions of the natural inclusion $*$, but they can also be viewed as "adjoints" of $*$, in the following sense (the lemma is an easy consequence of the above formulae):

Lemma 1.16. *For all $u \in NCP(e,1,n)$, for all $v \in NCP(e,e,n+1)$, we have*

$$v \preccurlyeq u_* \Leftrightarrow v^\sharp \preccurlyeq u \qquad \textit{and} \qquad u_* \preccurlyeq v \Leftrightarrow u \preccurlyeq v^\flat.$$

As expected, we have:

Lemma 1.17. *The poset $(NCP(e,e,n+1), \preccurlyeq)$ is a lattice.*

Proof. Once again, it is enough to check that $NCP(e,e,n+1)$ is stable by set-theoretical meet. Let $v, v' \in NCP(e,e,n+1)$. Let $v \wedge v'$ be their meet in $NCP_{\mu_{en}\cup\{0\}}$. We observe that $(v\wedge v')^\flat = v^\flat \wedge v'^\flat$, which is an element of $NCP(e,1,n)$. Thus $v \wedge v' \in NCP(e,e,n+1)$. ☐

1.6. Type $(1,1,n+1)$ intervals in $NCP(e,e,n+1)$.

Definition 1.18. *For $\zeta \in \mu_{en}$, we set*

$$a_\zeta := \{0, \zeta_{en}^1\zeta, \zeta_{en}^2\zeta, \ldots, \zeta_{en}^n\zeta\}.$$

and we consider the following asymmetric elements of $NCP(e,e,n+1)$:

- *the element m_ζ, the partition with the unique non-singleton part $\{0,\zeta\}$;*
- *the element M_ζ, made up of asymmetric part a_ζ, the remaining parts being of the form $\zeta' a_\zeta - \{0\}$, for $\zeta' \in \mu_e - \{1\}$.*

When $u, v \in NCP(e,e,n+1)$, we use the notation

$$[u,v] := \{w \in NCP(e,e,n+1) | u \preccurlyeq w \preccurlyeq v\}.$$

The following lemma characterises certain intervals in $NCP(e,e,n+1)$. We denote by res_ζ the restriction morphism from the lattice of partitions of $\mu_{en} \cup \{0\}$ to the lattice of partitions of a_ζ. Since a_ζ is a strictly convex $(n+1)$-gon, we have $NCP_{a_\zeta} \simeq NCP(1,1,n+1)$.

Lemma 1.19. (i) *The set of minimal (resp. maximal) asymmetric elements in $NCP(e,e,n+1)$ is $\{m_\zeta | \zeta \in \mu_{en}\}$ (resp. $\{M_\zeta | \zeta \in \mu_{en}\}$).*

(ii) *Let $\zeta \in \mu_{en}$. The map res_ζ induces a poset isomorphism*

$$\varphi_\zeta : ([\mathrm{disc}, M_\zeta], \preccurlyeq) \xrightarrow{\sim} (NCP_{a_\zeta}, \preccurlyeq).$$

(iii) *Let $\zeta \in \mu_{en}$. The map* res_ζ *induces a poset isomorphism*

$$\psi_\zeta : ([m_\zeta, \mathrm{triv}], \preccurlyeq) \xrightarrow{\sim} (NCP_{a_\zeta}, \preccurlyeq).$$

(iv) *The composition $\psi_\zeta^{-1}\varphi_\zeta$ is the map $u \mapsto m_\zeta \vee u$.*

Since a_ζ is convex and has cardinal $n+1$, we have $(NCP_{a_\zeta}, \preccurlyeq) \simeq (NCP(1,1,n+1), \preccurlyeq)$. It should also be noted that m_ζ is not finer than M_ζ, thus $[\mathrm{disc}, M_\zeta] \cap [m_\zeta, \mathrm{triv}] = \varnothing$.

Proof. (i) is clear.

(ii) Any part b of a given $u \in [\mathrm{disc}, M_\zeta]$ is either finer than a_ζ or finer than some $\zeta' a_\zeta - \{0\}$, $\zeta' \in \mu_e - \{1\}$. The symmetry conditions imply that given any $v \in NCP_{a_\zeta}$, there is a unique way to complete it with partitions of the $\zeta' a_\zeta - \{0\}$ to obtain an element of $[\mathrm{disc}, M_\zeta]$ (explicitly, the partition of $\zeta' a_\zeta - \{0\}$ is the restriction of $\zeta' v$).

(iii) Let $b_\zeta := a_\zeta \cup \{\zeta\}$. Elements of $[m_\zeta, \mathrm{triv}]$ are either asymmetric or long symmetric. Using the symmetry conditions, it is clear that they are uniquely determined by their restriction to b_ζ, and that 0 and ζ lie in the same part of this restriction. Conversely, any element of NCP_{b_ζ} satisfying the condition that 0 and ζ are connected may be obtained this way. It is also clear that the restriction from b_ζ to a_ζ identifies $\{v \in NCP_{b_\zeta} | 0 \text{ and } \zeta \text{ are in the same part}\}$ with NCP_{a_ζ}. This proves (iii).

It is clear that $u \mapsto m_\zeta \vee u$ induces a map $\theta : [\mathrm{disc}, M_\zeta] \to [m_\zeta, \mathrm{triv}]$. To obtain (iv), it is enough to check that, for all $u \in [\mathrm{disc}, M_\zeta]$, $\mathrm{res}_\zeta(u) = \mathrm{res}_\zeta \theta(u)$, which is straightforward. $\square$

1.7. The complement.

Definition 1.20. *Let $u, v \in NCP(e,e,n+1)$, with $u \preccurlyeq v$. We define an element $u \backslash v$, the* complement *of u in v, as follows:*

(A) *If u and v are both symmetric, we set*

$$u \backslash v := (u^\flat \backslash v^\flat)_*.$$

(B) *If v is asymmetric, we choose $\zeta \in \mu_{en}$ such that $v \preccurlyeq M_\zeta$. We set*

$$u \backslash v := \varphi_\zeta^{-1}(\varphi_\zeta(u) \backslash \varphi_\zeta(v)),$$

where the complement operation is defined on NCP_{a_ζ} via a standard identification with $NCP(1,1,n+1)$.

(C) *If u is asymmetric, we choose $\zeta \in \mu_{en}$ such that $m_\zeta \preccurlyeq u$. We set*

$$u \backslash v := \varphi_\zeta^{-1}(\psi_\zeta(u) \backslash \psi_\zeta(v)),$$

where the complement operation is defined on NCP_{a_ζ} via a standard identification with $NCP(1,1,n+1)$. (Note that we really mean φ^{-1} and not ψ^{-1}. This implies that $u \backslash v$ lies in $[\mathrm{disc}, M_\zeta]$).

For the above definition to make sense, one has to check that, in case (B) and (C), the result does not depend on the choice of ζ; also, when both u and v are asymmetric, one has to check that (B) and (C) give the same $u \backslash v$. We leave these to the reader to verify.

As it was the case in $NCP(1,1,n)$, this definition becomes more natural when interpreted in the associated braid group (see Proposition 2.4 below).

An essential property of the complement is that for all $u \preccurlyeq v$, one has

$$u \backslash v \preccurlyeq v.$$

Indeed, for the cases (A) and (B) of the definition, this follows immediately from the corresponding results for classical non-crossing partitions. In the situation (C), we have $\psi_\zeta(u) \backslash \psi_\zeta(v) \preccurlyeq \psi_\zeta(v)$. Since φ_ζ is a poset morphism, we have $u \backslash v \preccurlyeq \varphi_\zeta^{-1} \psi_\zeta(v)$. Looking at the construction of φ_ζ, one observes that, for any $w \in NCP_{a_\zeta}$, for any $\tilde{w} \in NCP(e,e,n+1)$ such that $\mathrm{res}_{a_\zeta}(\tilde{w}) = w$, one has $\varphi_\zeta^{-1}(w) \preccurlyeq \tilde{w}$. This applies in particular to $\tilde{w} = v$. Thus $u \backslash v \preccurlyeq \varphi_\zeta^{-1} \psi_\zeta(v) \preccurlyeq v$.

We also use the following notation:

Definition 1.21. *For all $u \in NCP(e,e,n+1)$, we set $\overline{u} := u \backslash \mathrm{triv}$.*

For example, since m_ζ is the minimal element of $[m_\zeta, \mathrm{triv}]$, $\psi_\zeta(m_\zeta)$ is the discrete element of NCP_{a_ζ}; clearly, $\psi_\zeta(\mathrm{triv})$ is the trivial element $\{a_\zeta\}$. Thus $\psi_\zeta(m_\zeta) \backslash \psi_\zeta(\mathrm{triv}) = \psi_\zeta(m_\zeta) \backslash \{a_\zeta\} = \{a_\zeta\}$, which maps to M_ζ by φ_ζ^{-1}. Thus

$$\overline{m_\zeta} = M_\zeta.$$

To compute M_ζ, there are several possible ζ' such that $m_{\zeta'} \preccurlyeq M_\zeta$. Take for example $\zeta' := \zeta_{en}\zeta$. Then $\psi_{\zeta'}(M_\zeta)$ is the partition of $a_{\zeta'}$ with two parts, $\{0, \zeta_{en}^2\zeta, \ldots, \zeta_{en}^n\zeta\}$ and $\{\zeta_{en}^{n+1}\zeta\}$. Thus $\psi_{\zeta'}(M_\zeta) \backslash \psi_{\zeta'}(\mathrm{triv}) = \psi_{\zeta'}(M_\zeta) \backslash \{a_{\zeta'}\}$ is the partition with $\{0, \zeta_{en}^{n+1}\zeta\}$ as only non-singleton part. The latter partition is mapped to $m_{\zeta_{en}^{n+1}\zeta} = \zeta_{en}^{n+1} m_\zeta$ by $\varphi_{\zeta'}$. We have proved that

$$\overline{M_\zeta} = \zeta_{en}^{n+1} m_\zeta.$$

As a consequence, for any m (resp. M) minimal (resp. maximal) asymmetric,

$$\overline{\overline{m}} = \zeta_{en}^{n+1} m \qquad \text{and} \qquad \overline{\overline{M}} = \zeta_{en}^{n+1} M.$$

We list below some basic properties of the complement:

Lemma 1.22. (i) *The map $u \mapsto \overline{u}$ is a poset anti-automorphism of $NCP(e,e,n+1)$.*
(ii) *If u is short symmetric, then $\overline{u}$ is long symmetric.*
(iii) *If u is long symmetric, then $\overline{u}$ is short symmetric.*
(iv) *If u is asymmetric, then $\overline{u}$ is asymmetric.*
(v) *For all u, $\overline{\overline{u}} = \zeta_e \zeta_{en} u$.*

Remark. By (ii) and (iii), the map $\phi : u \mapsto \overline{\overline{u}}$ is an automorphism of $NCP(e,e,n+1)$ whose order is the order of $\zeta_e\zeta_{en} = \zeta_{en}^{n+1}$, that is, $\frac{en}{(n+1)\wedge e}$. It is worth noting that there are several types of orbits for this automorphism:

- By definition, symmetric partitions are preserved under multiplication by μ_e, so the action of ϕ on symmetric partitions is by multiplication by ζ_{en}; this action has order n (the action is not free; e.g., disc and triv are fixed points).
- The group μ_{en} acts freely on the set of asymmetric partitions, which decomposes, under the action of ϕ, into orbits of equal cardinal $\frac{en}{(n+1)\wedge e}$.

1.8. **Height.** Let $u \in NCP(e, e, n+1)$. Let m be the number of parts of u. If u is symmetric, then parts of u come in orbits of cardinal e for the action of μ_e, except the isolated part $\{0\}$ (when u is short) and the long part (when u is long). In both symmetric cases, $m \equiv 1[e]$. When u is asymmetric, $e|m$.

We define the *height* $ht(u)$ of u as follows:

- If u is short symmetric, we set $ht(u) := n - (m-1)/e$.
- If u is asymmetric, we set $ht(u) := n + 1 - m/e$.
- If u is long symmetric, we set $ht(u) := n + 1 - (m-1)/e$.

It is not difficult to check the basic properties listed below:

Lemma 1.23. *Let $u \in NCP(e, e, n+1)$.*

(i) *If u is short symmetric, then $0 \leq ht(u) \leq n-1$.*
(ii) *If u is asymmetric, then $1 \leq ht(u) \leq n$.*
(iii) *If u is long symmetric, then $2 \leq ht(u) \leq n+1$.*
(iv) *For all $v \in NCP(e, e, n+1)$ such that $u \preccurlyeq v$, we have $ht(u) + ht(u \backslash v) = ht(v)$. In particular, $ht(u) + ht(\overline{u}) = n+1$.*
(v) *For all $v \in NCP(e, e, n+1)$, we have $u \preccurlyeq v \Rightarrow ht(u) \leq ht(v)$ and $u \prec v \Rightarrow ht(u) < ht(v)$.*
(vi) *For all $v \in NCP(e, e, n+1)$ such that $u \preccurlyeq v$, we may find a chain*

$$u = u_0 \preccurlyeq u_1 \preccurlyeq \cdots \preccurlyeq u_k = v$$

in $NCP(e, e, n+1)$ such that, for all i in $0, \ldots, k$, we have $ht(u_i) = ht(u) + i$.

2. Simple elements

2.1. **The complex reflection group $G(e, e, n+1)$.** View $\mathfrak{S}_{n+1}$ in its natural representation as the group of $(n+1) \times (n+1)$ permutation matrices. For any positive integer d, let $\Delta(de, e, n+1) \subset \mathrm{GL}_{n+1}(\mathbb{C})$ be the group of diagonal matrices with diagonal coefficients in μ_{de} and determinant in μ_d. The complex reflection group $G(de, e, n+1)$ is, by definition, the subgroup of $\mathrm{GL}_{n+1}(\mathbb{C})$ generated by $\mathfrak{S}_{n+1}$ and $\Delta(de, e, n+1)$ (see for example [BMR] for a general description of the classification of complex reflection groups). Then

$$G(de, e, n+1) \simeq \Delta(de, e, n+1) \rtimes \mathfrak{S}_{n+1}.$$

The group $G(e, e, n+1)$ has $e^n(n+1)!$ elements, among them are $e(n+1)(n+2)/2$ reflections, all with order 2 and forming a single conjugacy class. A minimal set of generating reflections has cardinal $n+1$: take for example the n permutation matrices $\sigma_i := (i \;\; i+1)$ (for $i = 1, \ldots, n$) plus the reflection $\mathrm{Diag}(\zeta_e, \overline{\zeta_e}, 1, 1, \ldots, 1, 1)\sigma_1$.

Let $X_0, \ldots, X_n$ be the canonical coordinates on $V := \mathbb{C}^{n+1}$ (we consider that matrices start with a 0-th line and a 0-th column). The reflecting hyperplanes have equations $X_i = \zeta X_j$, where $i, j \in \{0, \ldots, n\}$ and $\zeta \in \mu_e$. They form a single orbit under the action of $G(e, e, n+1)$.

It is sometimes useful to compare $G(e, e, n+1)$ with other groups in the infinite series of complex reflection groups. By definition, we have

$$G(e, e, n+1) \subset G(e, 1, n+1).$$

We may also construct a monomorphism

$$G(e, 1, n) \overset{\psi}{\hookrightarrow} G(e, e, n+1)$$

as follows. Let χ the character of $G(e,1,n)$ trivial on $\mathfrak{S}_n$ and coinciding with the determinant on $\Delta(e,1,n)$. Then, for any matrix $M \in G(e,1,r)$, the matrix (given by blocks)

$$\psi(M) := \begin{pmatrix} \chi(g) & 0 \\ 0 & M \end{pmatrix}$$

is in $G(e,e,n+1)$.

2.2. **The braid group** $B(e,e,n+1)$**.** Let $V := \mathbb{C}^{n+1}$, endowed with the action of $G(e,e,n+1)$. Let V^{reg} be the complement in V of the union of the reflection hyperplanes. By definition, the braid group $B(e,e,n+1)$ associated with $G(e,e,n+1)$ is the fundamental group of the quotient space $V^{\mathrm{reg}}/G(e,e,n+1)$. This definition involves the choice of a basepoint. Our preferred basepoint is chosen by means of Springer theory of regular elements ([Sp]).

The invariant degrees are $e, 2e, \dots, ne, n+1$. The largest degree, ne, is regular, in the sense of Springer (this means that there exists a element of $G(e,e,n+1)$ of order ne and having an eigenvector in V^{reg}; such an element is said to be *regular*). A typical regular element of order ne is

$$c := \begin{pmatrix} \overline{\zeta_e} & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & \zeta_e & 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

An example of regular eigenvector in $\ker(c - \zeta_{ne}\,\mathrm{Id})$ is

$$x_0 := \begin{pmatrix} 0 \\ \zeta_{ne} \\ \zeta_{ne}^2 \\ \vdots \\ \zeta_{ne}^n = \zeta_e \end{pmatrix}.$$

We choose x_0 as basepoint for V^{reg} and its orbit $\overline{x_0}$ as basepoint for the quotient space $V^{\mathrm{reg}}/G(e,e,n+1)$. Hence we fix the (now fully explicit) definition:

$$B(e,e,n+1) := \pi_1(V^{\mathrm{reg}}/G(e,e,n+1), \overline{x_0}).$$

Remarks.

- The initial symmetry between all reflecting hyperplanes is broken by the choice of c. This explains why our presentation involves two distinct types of generators.
- Note that $c \in \psi(G(e,1,n))$. The element c is actually the image via ψ of a maximal regular element in $G(e,1,n)$. A consequence of the main results in [BDM] is that we have a Garside monoid for $B(e,1,n)$, related to the choice of a maximal regular element, and admitting a description using planar partitions of μ_{en} invariant by μ_e.

2.3. **From non-crossing partitions to elements of** $B(e,e,n+1)$ **and** $G(e,e,n+1)$**.** We construct here a map $NCP(e,e,n+1) \to B(e,e,n+1)$.

We work with a given $u \in NCP(e,e,n+1)$. To simplify notations, we sometimes omit referring to u, though the construction of course depends on u.

For every $z \in \mu_{ne} \cup \{0\}$, denote by z' the successor of z for the counterclockwise ordering of the part of u containing z (when u is long symmetric, when dealing with the long part, which is not convex, we consider the counterclockwise ordering of the non-zero elements of this part, and set $0' := 0$).

We consider a path $\gamma_{u,z}$ (also denoted by γ_z if there is no ambiguity on u) as follows:

- If the part of u containing z contains three or more elements, we set

$$\begin{array}{rcl} \gamma_z : [0,1] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & (1-t)z + tz' \end{array}$$

- If the part of u containing z contains exactly two elements (z and z'), we slightly perturbate the above definition by setting

$$\begin{array}{rcl} \gamma_z : [0,1] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & (1-t)z + tz' + it(1-t)(z-z')\varepsilon_n \end{array}$$

  where i is the standard square root of -1 used to fix the orientation of $\mathbb{C}$, and $\varepsilon_n \in \mathbb{R}_{>0}$ is a fixed "small enough" number, depending only on n.
- If z is isolated in u, we take for γ_z the constant path $t \mapsto z$.

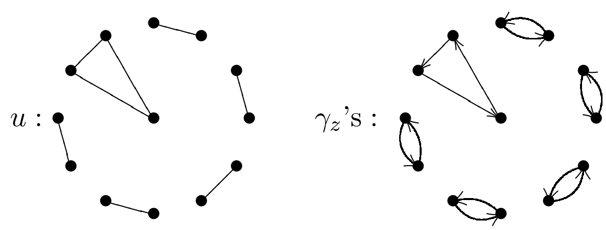

FIGURE 7. The various γ_z's, for a given $u \in NCP(3,3,5)$.

We use these paths in $\mathbb{C}$ to associate to u a path in V^{reg}, with initial point x_0. When u is asymmetric, there is unique $z \in \mu_{ne}$ such that $z' = 0$ (in the example illustrated above, $z = \zeta_{12}^5$). We say that $j \in \{1,\dots,n\}$ is *special* (with respect to u) if u is asymmetric and j is such that $z \in \mu_e \zeta_{ne}^j$, for the unique z such that $z' = 0$ (when u is asymmetric, this condition selects a unique $j \in \{1,\dots,n\}$; when u is symmetric, there are no special integers; in our above example, the special integer is 1).

If $j \in \{1,\dots,n\}$ is special, we set $\gamma_j := \zeta^{-1}\gamma_z$, where z is the above predecessor of 0, and ζ is the element of μ_e such that $z = \zeta\zeta_{ne}^j$. If j is not special, we set $\gamma_j := \gamma_{\zeta_{ne}^j}$. The path γ_u is defined

by

$$\begin{array}{rcl} \gamma_u : [0,1] & \longrightarrow & \mathbb{C}^n \\ t & \longmapsto & \begin{pmatrix} \gamma_0(t) \\ \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{pmatrix} \end{array}$$

Figure 8 illustrates this construction.

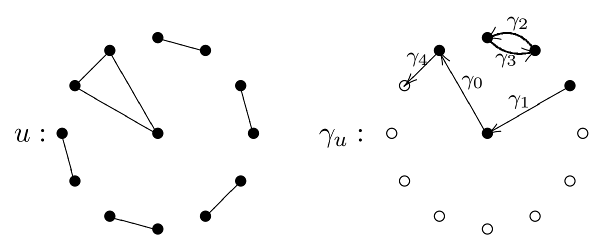

FIGURE 8. Illustration of the map $u \mapsto \gamma_u$; the initial positions of the γ_j's are marked by black dots.

Provided that ε_n is taken close enough to 0, this indeed defines a path in V^{reg}, whose homotopy class does not depend on the explicit ε_n.

The final endpoint $\gamma_u(1)$ of γ_u lies in the orbit $\overline{x_0}$. This follows from the following easy lemma, left to the reader:

Lemma 2.1. *A vector $(z_0, \dots, z_n) \in \mathbb{C}^{n+1}$ is in the orbit $\overline{x_0}$ if and only if we have $\{z_0, \dots, z_n\} = X \cup \{0\}$, where $X \subset \mu_{ne}$ is a set of coset representatives of μ_{ne}/μ_e.*

Thus the path γ_u defines a loop $\overline{\gamma_u}$ in the quotient $V^{\text{reg}}/G(e,e,n+1)$.

Definition 2.2. *We denote by b_u the element of $B(e,e,n+1)$ represented by the loop $\overline{\gamma_u}$. We set*

$$P_B := \{b_u | u \in NCP(e,e,n+1)\}.$$

As in [BMR], we have a fibration exact sequence

$$1 \longrightarrow P(e,e,n+1) \longrightarrow B(e,e,n+1) \xrightarrow{\pi} G(e,e,n+1) \longrightarrow 1,$$

where $P(e,e,n+1) = \pi_1(V^{\text{reg}}, x_0)$. The interpretation of the morphism π is as follows: given $b \in B(e,e,n+1)$, we may represent it by a loop in $V^{\text{reg}}/G(e,e,n+1)$ with endpoint $\overline{x_0}$; this loops lifts to a unique path in V^{reg} with initial point x_0; the final endpoint x of this path is in the orbit $\overline{x_0}$; the element $\pi(b)$ is the unique $g \in G(e,e,n+1)$ such that $g(x_0) = x$.

Definition 2.3. *For all $u \in NCP(e,e,n+1)$, we set $g_u := \pi(b_u)$. We set*

$$P_G := \pi(P_B) = \{g_u | u \in NCP(e,e,n+1)\}.$$

So far we have constructed a commutative diagram

$$\begin{tikzcd} & P_B \arrow[dd, "\pi"] \\ NCP(e,e,n+1) \arrow[ur, "u\mapsto b_u"] \arrow[dr, "u\mapsto g_u"'] & \\ & P_G \end{tikzcd}$$

Our task in the next sections will be to prove that these maps are bijective – actually, P_G will be given a more intrinsic definition and a natural poset structure, making $u \mapsto g_u$ a poset isomorphism.

We may endow P_B with the *left divisibility partial order*, again denoted by $\preccurlyeq$ and defined by

$$\forall b, b' \in P_B, b \preccurlyeq b' \overset{\text{def}}{\iff} \exists b'' \in P_B, bb'' = b'.$$

In this setting, a consequence of the next proposition is that the map $u \mapsto b_u$ is a poset morphism (it will later be proved to be a poset isomorphism).

Proposition 2.4. *For any $u, v \in NCP(e,e,n+1)$ such that $u \preccurlyeq v$, we have $b_u b_{u\backslash v} = b_v$.*

Note that we have already seen a result of this nature, for the type A situation (see Figure 4). Figure 9 illustrates the relation with an example in $B(3,3,5)$. The paths γ_u and $\gamma_{u\backslash v}$ may not be concatenated in V^{reg}; one insteads concatenates γ_u (whose final point is $g_u(x_0)$) with the transformed path $g_u\gamma_{u\backslash v}$.

Proof. We have to deal with the three cases of Definition 1.20.

(A) In [BDM], $NCP(e,1,n)$ was interpreted in relation with the braid group $B(e,1,n)$ of $G(e,1,n)$ (the latter being viewed as the centraliser of the e-th root of the full twist in $G(1,1,en)$). It is easy to see that our definition of $u\backslash v$ and b_u, for symmetric elements of $NCP(e,e,n+1)$, coincide via $\flat$ with the corresponding constructions from *loc. cit.*. The desired result is then a consequence of *loc. cit.*, Proposition 1.8 (ii) (the element there denoted by δ_λ is our b_u). Another way to recover the case (A), without using [BDM], is by a direct check using Sergiescu relations.

(B) Assume v is asymmetric, and choose ζ such that $v \preccurlyeq M_\zeta$. Instead of representing b_v by the path γ_v, which starts at x_0, we may replace γ_u by any $g(\gamma_u)$ with $g \in G(e,e,n+1)$. Thanks to Lemma 2.1, we may choose g such that

$$g(x_0) = \begin{pmatrix} 0 \\ \zeta_{en}^1\zeta \\ \vdots \\ \zeta_{en}^n\zeta \end{pmatrix}.$$

A direct computation shows that

$$g(\gamma_v) = \begin{pmatrix} \gamma_{v,0} \\ \gamma_{v,\zeta_{en}^1\zeta} \\ \vdots \\ \gamma_{v,\zeta_{en}^n\zeta} \end{pmatrix}.$$

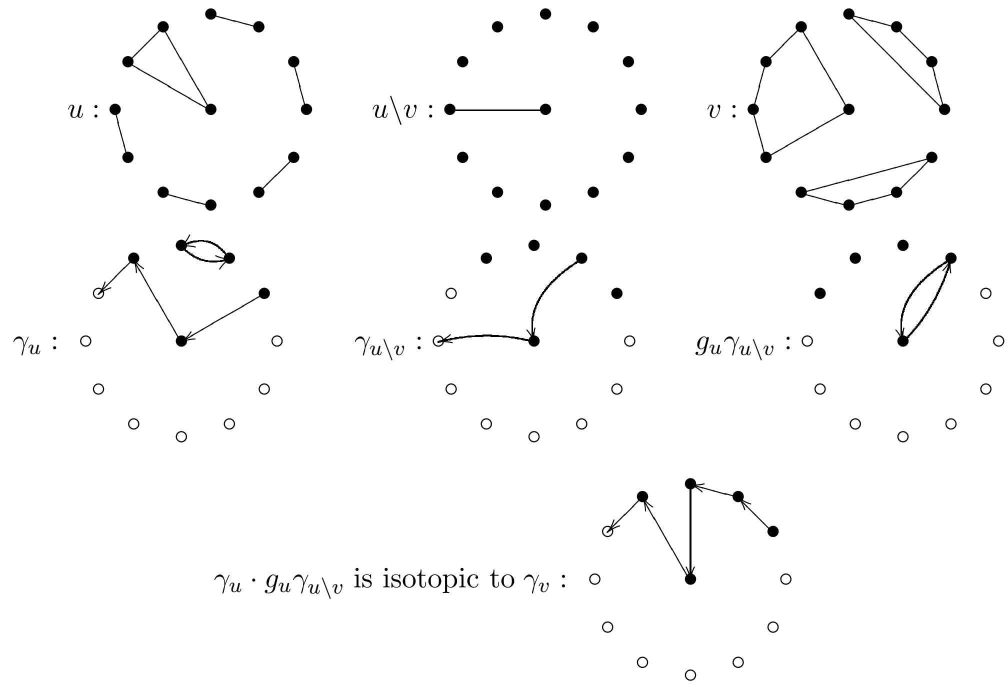

FIGURE 9. Illustration of the relation $b_u b_{u\backslash v} = b_v$ on an example. The black dots indicate the coordinates at the starting point of the paths. The product $b_u b_{u\backslash v}$ is represented in V^{reg} by the path obtained by concatenating γ_u and $g_u \cdot \gamma_{u\backslash v}$; this concatenation is homotopic to γ_v.

Since $v \preccurlyeq M_\zeta$, we have

$$\{\gamma_{v,0}(0), \gamma_{v,\zeta_{en}^1\zeta}(0), \dots, \gamma_{v,\zeta_{en}^n\zeta}(0)\} = \{\gamma_{v,0}(1), \gamma_{v,\zeta_{en}^1\zeta}(1), \dots, \gamma_{v,\zeta_{en}^n\zeta}(1)\} = a_\zeta.$$

In particular, the final endpoint $g(\gamma_v)(1)$ is related to the starting point $g(\gamma_v)(0)$ by a permutation matrix $\sigma \in G(1,1,n+1)$. In other words, $g(\gamma_v)$ may be viewed as a classical type A braid involving $n+1$ strings, with initial and final position at a_ζ. A similar discussion applies to u. In this classical type A braid group, there is a complement operation and the relation similar to the one we want to prove here (see Figure 4, or [BDM]). Definition 1.20 is precisely designed to be compatible with this classical relation: the classical relation asserts that two braids are homotopic; one may easily sees that the homotopy may be chosen so that at all time, all strings lie in the convex hull of a_ζ; this implies that the desired homotopy holds in V^{reg}.

We leave the details to the reader, as well as the proof in case (C), which may be carried out in a similar manner. □

3. NON-CROSSING PARTITIONS OF HEIGHT 1 AND LOCAL GENERATORS

In this section, we have a closer look at the image by the previous maps of the set of non-crossing partitions of height 1. The image by $u \mapsto b_u$ will be the generating set in our new presentation for $B(e,e,n+1)$.

Definition 3.1. *Let $p,q \in \mathbb{Z}$ with $0 < |p-q| < n$. We set $u_{p,q}$ to be the (short symmetric) element of $NPC(e,e,n+1)$ whose non-singleton parts are*

$$\{\zeta_{ne}^p, \zeta_{ne}^q\}\ ,\ \zeta_e\{\zeta_{ne}^p, \zeta_{ne}^q\}\ ,\ \zeta_e^2\{\zeta_{ne}^p, \zeta_{ne}^q\}\ ,\ \ldots\ ,\ \zeta_e^{e-1}\{\zeta_{ne}^p, \zeta_{ne}^q\}.$$

We extend this notation to the situation where p and q are the images in $\mathbb{Z}/en\mathbb{Z}$ of integers $\tilde{p}$ and $\tilde{q}$ with $0 < |\tilde{p}-\tilde{q}| < n$.

For all $p \in \mathbb{Z}$, we set u_p to be the (asymmetric) element of $NCP(e,e,n+1)$ with only non-singleton part $\{0, \zeta_{en}^p\}$. We extend this notation to the situation where $p \in \mathbb{Z}/en\mathbb{Z}$.

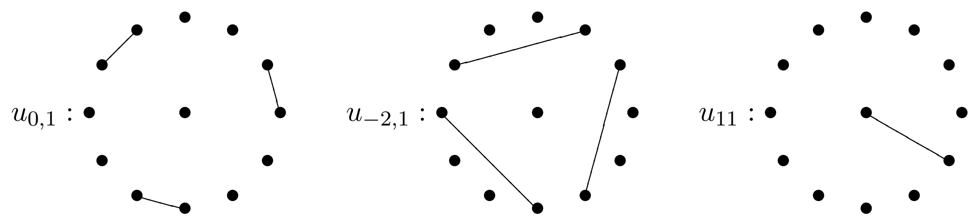

FIGURE 10. Some height 1 elements of $NCP(3,3,5)$

Lemma 3.2. *There are $en(n-1)$ unordered pairs of elements $p,q \in \mathbb{Z}/en\mathbb{Z}$ who have representatives $\tilde{p},\tilde{q} \in \mathbb{Z}$ with $0 < |\tilde{p}-\tilde{q}| < n$. For all such p,q, we have $u_{p,q} = u_{p+n,q+n}$. Conversely, if $u_{p',q'} = u_{p,q}$, then $p' = p+kn, q' = q+kn$ or $p' = q+kn, q' = p+kn$ for some k. Thus there are $n(n-1)$ distinct partitions of type $u_{p,q}$.*

There are en distinct partitions of type u_p.

The $n(n-1)$ elements of the form $u_{p,q}$, together with the en elements of the form $u_p^{(i)}$, is a complete list of height 1 elements of $NCP(e,e,n+1)$.

In view of Lemma 1.23 (v) and (vi), minimal height 1 non-crossing partitions are precisely minimal non-discrete elements in $NCP(e,e,n+1)$.

Definition 3.3. *With p,q corresponding to the situations considered in Definition 3.1, we set*

$$a_{p,q} := b_{u_{p,q}} \qquad \text{and} \qquad a_p := b_{u_p}.$$

We set $A := \{a_{p,q} | p,q \in \mathbb{Z}, 0 < |p-q| < n\} \cup \{a_p | p \in \mathbb{Z}/en\mathbb{Z}\}$.

Since all reflections in $G(e,e,n+1)$ have order 2, the notions of *visible hyperplanes*, *local generators* and *local monoids* may be imported at no cost from section 3 of [B1] to our context. The following proposition states that the geometric interpretation of the dual braid monoid carries on to our situation. It is not difficult to prove the following result – this is just a matter of basic affine geometry – but since we do not use it later we do not include a proof.

Proposition 3.4. *The elements of A are braid reflections (aka "meridiens" or "generators-of-the-monodromy"). The corresponding reflecting hyperplanes are visible from our basepoint x_0, and A is actually the set of local generators at x_0. No other hyperplanes are visible from x_0. Hence A generate the local braid monoid at x_0.*

For finite real reflection groups, among local generators, one may always find "classical generators" (corresponding to an Artin presentation); in particular, local generators indeed generate the braid group ([B1], Propositions 3.4.3(1) and 3.4.5). In our situation, one may wonder whether a Broué-Malle-Rouquier generating set may be found inside A. This may be done; however first, for the convenience of the reader, let us quote from [BMR] their result about $B(e, e, n+1)$:

Theorem 3.5 (Broué-Malle-Rouquier). *The group $B(e, e, n+1)$ admits the following presentation:*
Generators: $\tau_2, \tau_2', \tau_3, \dots, \tau_n, \tau_{n+1}$
Relations: *The commuting relations are: $\tau_i\tau_j = \tau_j\tau_i$ whenever $|i-j| \geq 2$, together with $\tau_2'\tau_j = \tau_j\tau_2'$ for all $j \geq 4$. The others are:*

$$\begin{array}{rcll} \tau_i\tau_j & = & \tau_j\tau_i & \text{for } |i-j| \geq 2 \\ \tau_2'\tau_j & = & \tau_j\tau_2' & \text{for } j \geq 4 \\ \langle \tau_2\tau_2' \rangle^e & = & \langle \tau_2'\tau_2 \rangle^e & \\ \tau_i\tau_{i+1}\tau_i & = & \tau_{i+1}\tau_i\tau_{i+1} & \text{for } i = 2, \dots, n \\ \tau_2'\tau_3\tau_2' & = & \tau_3\tau_2'\tau_3 & \\ \tau_3\tau_2\tau_2'\tau_3\tau_2\tau_2' & = & \tau_2\tau_2'\tau_3\tau_2\tau_2'\tau_3 & \end{array}$$

where the expression $\langle ab \rangle^k$ denotes the alternating product $aba\cdots$ of length k.

Broué-Malle-Rouquier construct their generators (which are braid reflections) in an inductive manner, using suitable fibrations. The actual generators depend on a certain number of choices, and by having a careful look at the proof in [BMR], it is not difficult to rephrase the above theorem in the more precise manner:

Theorem 3.6 (after Broué-Malle-Rouquier). *Denote by BMR the abstract group presented by the above Broué-Malle-Rouquier presentation. There is an isomorphism*

$$\begin{array}{rcll} BMR & \xrightarrow{\sim} & B(e, e, n+1) & \\ \tau_2 & \longmapsto & a_0 & \\ \tau_2' & \longmapsto & a_n & \\ \tau_i & \longmapsto & a_{i-3,i-2} & \text{for } i = 3, \dots, n+1. \end{array}$$

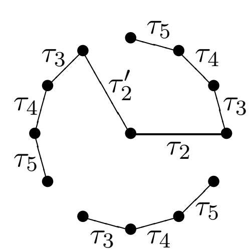

FIGURE 11. Typical Broué-Malle-Rouquier generators for $B(3, 3, 5)$

We complete this section with a computation of the reflecting hyperplanes of the non-crossing reflections. In view of Lemma 3.2, it is not difficult to see that the the considered indices fully parametrise the height 1 non-crossing partitions.

Lemma 3.7. *Let $p,q,\in \mathbb{Z}$.*

If $1 \leq p < q \leq n$, then $g_{u_{p,q}}$ is the reflection with hyperplane $X_p = X_q$, and $g_{u_{q,p+n}}$ is the reflection with hyperplane $\zeta_e X_p = X_q$.

If $1 \leq p \leq en$, write $p = in + j$, with $j \in \{1,\dots,n\}$. Then g_{u_p} is the reflection with hyperplane $X_0 = \zeta_e^i X_j$.

Proof. The computation goes as explained in Subsection 2.4, from where we take our notation.

If $1 \leq p < q \leq n$, the final point of $\gamma_{u_{p,q}}$ is

$$(0, \zeta_{en}, \dots, \zeta_{en}^{p-1}, \zeta_{en}^{q}, \zeta_{en}^{p+1}, \dots, \zeta_{en}^{q-1}, \zeta_{en}^{p}, \zeta_{en}^{q+1}, \dots, \zeta_{en}^{n}).$$

Thus $g_{u_{p,q}}$ is the permutation matrix associated with the transposition $(p\ q)$; in other words, it is the reflection with hyperplane $X_p = X_q$. The final point of $\gamma_{u_{q,p+n}}$ is

$$(0, \zeta_{en}, \dots, \zeta_{en}^{p-1}, \zeta_e^{-1}\zeta_{en}^{q}, \zeta_{en}^{p+1}, \dots, \zeta_{en}^{q-1}, \zeta_e\zeta_{en}^{p}, \zeta_{en}^{q+1}, \dots, \zeta_{en}^{n}).$$

The element of $G(e,e,n+1)$ sending x_0 to this final point resembles the permutation matrix of $(p\ q)$, except that the submatrix of lines p,q and columns p,q is

$$\begin{pmatrix} 0 & \zeta_e^{-1} \\ \zeta_e & 0 \end{pmatrix}$$

This is the reflection with hyperplane $\zeta_e X_p = X_q$.

If $i \in \{0,\dots,e-1\}$, $j \in \{1,\dots,n\}$, the final point of $\gamma_{u_{in+j}}$ is

$$(\zeta_{en}^{in+j}, \zeta_{en}, \dots, \zeta_{en}^{j-1}, 0, \zeta_{en}^{j+1}, \dots, \zeta_{en}^{n}).$$

The associated matrix resembles the permutation matrix of $(0\ p)$, except that the submatrix of lines $0,j$ and columns $0,j$ is

$$\begin{pmatrix} 0 & \zeta_e^{i} \\ \zeta_e^{-i} & 0 \end{pmatrix}$$

(to understand where the ζ_e^{-i} comes from, remember that the product of the non-zero coefficients must be 1). This is the reflection with hyperplane $X_0 = \zeta_e^i X_j$. $\square$

4. A length function on $G(e,e,n+1)$

Let T be the set of all reflections in $G(e,e,n+1)$. As in Section 1 of [B1], we consider, for each $g \in G(e,e,n+1)$, the set $\mathrm{Red}_T(g)$ of *reduced T-decompositions* of g, *i.e.*, of minimal length sequences $(t_1,\dots,t_k)$ such that $g = t_1 \dots t_k$, and we denote by $l_T(g)$ the length of such minimal sequences. We also have a partial order relation $\preccurlyeq_T$: for all $g,h \in G(e,e,n+1)$, $g \preccurlyeq_T h$ if and only if there exists $(t_1,\dots,t_k) \in \mathrm{Red}_T(h)$ and $l \leq k$ such that $g = t_1 \dots t_l$. (Since T is a union of conjugacy classes, one obtains an equivalent definition when replacing the condition $g = t_1 \dots t_l$ by $g = t_{i_1} \dots t_{i_l}$, for some increasing sequence $1 \leq i_1 < i_2 < \dots < i_l \leq k$.)

For obvious reasons from elementary linear algebra, we have a first approximation for the function l_T:

$$\forall g \in G(e,e,n+1), l_T(g) \geq \mathrm{codim}(\ker(g-1)).$$

A major difference with the situation with real reflection groups is that the above inequality may indeed be strict, as we will see shortly. The analog of the basic Lemma 1.3.1 of [B1] does not hold for all g but, as it will appear shortly, many results from *loc. cit.* continue to hold.

In the following lemma, c is the element defined in Subsection 2.2.

Lemma 4.1. (i) *We have* $c = g_{\mathrm{triv}}$.
(ii) *For all* $u \in NCP(e,e,n+1)$, *we have* $ht(u) = \mathrm{codim}(\ker(g_u - 1)) = l_T(g_u)$.
(iii) *For all* $u \in NCP(e,e,n+1)$, *we have*

$$\ker(g_u - 1) = \bigcap_{v \preccurlyeq u, ht(v)=1} \ker(g_v - 1).$$

Proof. Part (i) is obtained directly from the definitions.

Note that the case $ht(u) = 1$ of (ii) follows from the fact that g_u is then a reflection.

Let $u \in NCP(e,e,n+1)$. We prove first the inequality $ht(u) \geq l_T(u)$ from (ii). By Lemma 1.23(vi), we may find a "filtration" $u_0 \preccurlyeq u_1 \preccurlyeq u_2 \preccurlyeq \cdots \preccurlyeq u_{ht(u)}$ with $u_{ht(u)} = u$ and $\forall i, ht(u_i) = i$. For $i = 1, \ldots, ht(u)$, set $v_i := u_{i-1} \backslash u_i$. By Lemma 1.23(iv), we have $ht(v_i) = 1$ and, as we have seen above, g_{v_i} is a reflection. Since

$$g_u = g_{u_{ht(u)}} = g_{u_{ht(u)-1}} g_{v_{ht(u)}} = g_{u_{ht(u)-2}} g_{v_{ht(u)-1}} g_{v_{ht(u)}} = \cdots = g_{v_1} g_{v_2} \ldots g_{v_{ht(u)}},$$

we have a T-decomposition of length $ht(u)$ of u.

The case $u = \mathrm{triv}$ of (ii) is a direct check.

We may now prove (ii) in full generality. For any $u \in NCP(e,e,n+1)$, we have $c = g_u g_{\overline{u}}$ (by Proposition 2.4) and $ht(c) = ht(u) + ht(\overline{u})$ (by Lemma 1.23(iv)). Thus

$$ht(u) + ht(\overline{u}) = ht(c) = \mathrm{codim}(\ker(c-1)) = \mathrm{codim}(\ker(g_u g_{\overline{u}} - 1)).$$

By basic linear algebra, $\ker(g_u g_{\overline{u}} - 1) \supseteq \ker(g_u - 1) \cap \ker(g_{\overline{u}} - 1)$ and $\mathrm{codim}(\ker(g_u - 1) \cap \ker(g_{\overline{u}} - 1)) \leq \mathrm{codim}(\ker(g_u - 1)) + \mathrm{codim}(\ker(g_{\overline{u}} - 1))$. Thus

$$ht(u) + ht(\overline{u}) \leq \mathrm{codim}(\ker(g_u - 1)) + \mathrm{codim}(\ker(g_{\overline{u}} - 1)).$$

But on the other hand, we already checked that

$$ht(u) \geq l_T(g_u) \geq \mathrm{codim}(\ker(g_u - 1))$$

and

$$ht(\overline{u}) \geq l_T(g_{\overline{u}}) \geq \mathrm{codim}(\ker(g_{\overline{u}} - 1)).$$

All considered inequalities must be equalities, and so (ii) follows.

Note that, given as earlier a "filtration" $u_0 \preccurlyeq u_1 \preccurlyeq u_2 \preccurlyeq \cdots \preccurlyeq u_{ht(u)}$ with $u_{ht(u)} = u$ and $\forall i, ht(u_i) = i$, the same basic linear algebra considerations, together with (ii), imply that $\ker(g_u - 1) = \bigcap_{i=1}^n \ker(g_{v_i} - 1)$ (where $v_i := u_{i-1}^{-1} u_i$). Since, by Lemma 1.23(vi), we may start this filtration with any partition u_1 of height 1 such that $u_1 \preccurlyeq u$, we obtain (iii). □

Proposition 4.2. *Let* $u, v \in NCP(e,e,n+1)$. *The following assertions are equivalent:*
(i) $u \preccurlyeq v$,
(ii) $g_u \preccurlyeq_T g_v$,
(iii) $\ker(g_u - 1) \supseteq \ker(g_v - 1)$.

Proof. Assuming (i), one has $g_u g_{u \backslash v} = g_v$, with $l_T(g_u) + l_T(g_{u\backslash v}) = ht(u) + ht(u \backslash v) = ht(v) = l_T(g_v)$ (using Lemma 4.1 (ii) and Lemma 1.23 (iv)), thus (ii).

Assuming (ii), we have $(t_1, \ldots, t_k)\, \mathrm{Red}_T(g_v)$ and $l \leq k$ such that $(t_1, \ldots, t_l) \in \mathrm{Red}_T(g_u)$. We have $\cap_{i=1}^k \ker(t_i - 1) \subseteq \ker(g_u - 1)$. A priori, $\mathrm{codim}(\cap_{i=1}^k \ker(t_i - 1)) \leq k$. By Lemma 4.4 (ii), we know that $\mathrm{codim}(\ker(g_u - 1)) = k$. Thus $\cap_{i=1}^k \ker(t_i - 1) = \ker(g_u - 1)$. Similarly, $\cap_{i=1}^l \ker(t_i - 1) = \ker(g_v - 1)$. This gives (iii).

To prove (iii) $\Rightarrow$ (i), we first observe that one may use Lemma 3.7 and Lemma 4.1 to find an explicit description of $\ker(g_v - 1)$, as follows.

Let

$$B_v := \{ \bigcup_{\zeta \in \mu_e} \zeta a | a \text{ part of } v \}.$$

Each element of B_v is a union of parts of v. Clearly, B_v is a (possibly crossing) partition of $\mu_{en} \cup \{0\}$, with $v \preccurlyeq B_v$. To any part a of B_v, we assign an integer i_a as follows. Since a is closed under μ_e-action, it intersects $\{0, \zeta_{en}, \zeta_{en}^2, \ldots, \zeta_{en}^n\}$ (which is a fundamental domain for the μ_e-action on $\mu_{en} \cup \{0\}$). If $0 \in a$, set $i_a := 0$. If $0 \notin a$, choose $i_a \in \{1, \ldots, n\}$ such that $\zeta_{en}^{i_a} \in a$. If v is short symmetric or asymmetric, we set

$$I_v := \{ i_a | a \text{ part of } B_v \};$$

if v is long symmetric, we set

$$I_v := \{ i_a | a \text{ part of } B_v, 0 \notin a \}.$$

Claim. *We have* $|I_v| = \dim(\ker(g_v - 1))$. *For any function* $\alpha : I_v \to \mathbb{C}$, *there exists a unique* $x \in \ker(g_v - 1)$ *such that for all* $i \in I_v$, *the* i-*th coordinate of* x *is* $\alpha(i)$.

The first statement of the claim follows easily from a case-by-case analysis of the μ_e action on the set of parts of v, from the defining formula for $ht(v)$, and from the relation $ht(v) = \operatorname{codim}(\ker(g_v - 1))$ (Lemma 4.1 (ii)).

To prove the second statement, we observe that for any $i, j \in \{1, \ldots, n\}$ such that ζ_{en}^i and ζ_{en}^j lie in the same part of B_v, then ζ_{en}^i and $\zeta \zeta_{en}^j$ lie in the same part of v for some $\zeta \in \mu_e$, and $\ker(g_v - 1)$ is contained in the corresponding hyperplane, of equation $X_i = \zeta' X_j$ for some $\zeta' \in \mu_e$ (use Lemma 4.1 (iii) and Lemma 3.7). For the part a_0 of B_v containing 0, we similarly remark that $\ker(g_v - 1)$ is included in hyperplanes of equations $X_0 = \zeta'' X_i$, for each $i \in \{1, \ldots, n\}$ such that $\zeta_{en}^i \in a_0$. When v is long symmetric, we note in addition that the long part contains both $0, \zeta_{en}^i, \zeta_{en}^{i+n}$ for some i, thus we have (once again by Lemma 4.1 (iii) and Lemma 3.7) relations $X_0 = X_i = \zeta_e X_i$, and $X_0 = 0$. Altogether, these observations imply that any $x \in \ker(g_v - 1)$ is entirely determined by its I_v coordinates. One concludes using the relation $|I_v| = \dim(\ker(g_v - 1))$.

We are now ready to prove (iii) $\Rightarrow$ (i). Assume that (i) does not hold. Choose a generic x in $\ker(g_v - 1)$. Since u is not finer than v, we may find $\zeta, \zeta' \in \mu_{en} \cup \{0\}$ lying in the same part of u, but not in the same part of v. Up to multiplying by an element of μ_e, we may assume that $\zeta \in \{0, \zeta_{en}, \ldots, \zeta_{en}^n\}$. Set $i \in \{0, \ldots, n\}$ such that $\zeta = \zeta_{en}^i$ or $i = 0 = \zeta$. Using once again Lemma 4.1 (iii) and Lemma 3.7, one may find an equation $X_i = \zeta' X_j$, not satisfied by x, and such that $\ker(g_u - 1)$ must be included in the corresponding hyperplane; hence (iii) may not hold. $\square$

The above proposition tells us about T-divisibility among elements of P_G. Can an element of $G(e, e, n+1)$ not associated to a non-crossing partition divide an element of P_G? We will prove below that this may not happen. We start with the easier case of reflections.

Lemma 4.3. *Let* $t \in T$. *Then* $t \in P_G \Leftrightarrow t \preccurlyeq_T c$.

Proof. The $\Rightarrow$ implication follows from Proposition 4.2. To prove the converse, we proceed by direct computation. The reflections not in P_G have hyperplanes of equation $\zeta X_i = X_j$, with $i, j \in \{1, \ldots, n\}$, $i < j$ and $\zeta \in \mu_e - \{1, \zeta_e\}$ (see the explicit list of reflections in P_G from Lemma 3.7).

It is convenient to introduce condensed notation to describe monomial matrices: if M is an $(n+1)\times(n+1)$ monomial matrix (with lines and columns indexed by $\{0,\dots,n\}$), we represent M by

$$\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ \alpha_1 & \alpha_2 & \dots & \alpha_p \end{pmatrix}\begin{pmatrix} j_1 & j_2 & \dots & j_q \\ \beta_1 & \beta_2 & \dots & \beta_q \end{pmatrix}\begin{pmatrix} k_1 & k_2 & \dots & k_r \\ \gamma_1 & \gamma_2 & \dots & \gamma_r \end{pmatrix}\dots$$

where

$$\begin{pmatrix} i_1 & i_2 & \dots & i_p \end{pmatrix}\begin{pmatrix} j_1 & j_2 & \dots & j_q \end{pmatrix}\begin{pmatrix} k_1 & k_2 & \dots & k_r \end{pmatrix}\dots$$

is the cycle decomposition of the underlying permutation σ_M (including singleton cycles), and the second line gives corresponding non-trivial coefficient: for example, if $n=3$, by

$$\begin{pmatrix} 0 \\ \zeta \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ \lambda & \mu & \nu \end{pmatrix},$$

we mean the matrix

$$\begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \end{pmatrix}$$

The order of the factors does not matter. Each factor is called a *generalised cycle.*

With this notation system, c is represented by

$$\begin{pmatrix} 0 \\ \zeta_e^{-1} \end{pmatrix}\begin{pmatrix} n & n-1 & \dots & 2 & 1 \\ 1 & 1 & \dots & 1 & \zeta_e \end{pmatrix}$$

and the reflection t in $G(e,e,n+1)$ with hyperplane $X_i = \zeta X_j$ $(1 \leq i < j \leq n)$ is represented by

$$\begin{pmatrix} i & j \\ \zeta & \zeta^{-1} \end{pmatrix}$$

The rule for multiplying such symbols is easy to figure out. The product tc is represented by

$$\begin{pmatrix} 0 \\ \zeta_e^{-1} \end{pmatrix}\begin{pmatrix} n & n-1 & \dots & j+2 & j+1 & i & i-1 & \dots & 1 \\ 1 & 1 & \dots & 1 & \zeta^{-1} & 1 & 1 & \dots & \zeta_e \end{pmatrix}\begin{pmatrix} j & j-1 & \dots & i+2 & i+1 \\ 1 & 1 & \dots & 1 & \zeta \end{pmatrix}$$

We have to prove that if $\zeta \notin \{1,\zeta_e\}$, the kernel of $tc-1$ is trivial. This follows from the elementary remark: a monomial matrix $M \in \mathrm{GL}(\mathbb{C}^{n+1})$ has some non-trivial fixed points if and only if it has a generalised factor whose product of coefficients is 1 (actually, the number of such factors is $\dim(\ker(M-1))$) – in our situation, we have three generalised cycles with product of coefficients ζ_e^{-1}, $\zeta^{-1}\zeta_e$ and ζ. □

Proposition 4.4. *For all $g \in G(e,e,n+1)$, we have*

$$g \in P_G \quad \Leftrightarrow \quad g \preccurlyeq_T c.$$

In the proof (and later in the text), we use the notion of *parabolic subgroup* of a reflection group: if $W \subseteq \mathrm{GL}(V)$, if $X \subseteq V$, the associated parabolic subgroup W_X is $\{w \in W | \forall x \in X, w(x) = x\}$.

Proof. The implication $\Rightarrow$ follows from Proposition 4.2, (i) $\Rightarrow$ (ii), and from the fact that $c = g_{\mathrm{triv}}$.

We prove the converse implication by induction on n. Let $g \in G(e,e,n+1)$ such that $g \preccurlyeq_T c$. Let $t \in T$ such that $t \preccurlyeq_T g$. Set $g' := t^{-1}g$, $c' := t^{-1}c$. Since $t \preccurlyeq_T g \preccurlyeq_T c$, we deduce from Lemma 4.3 that $t \in P_G$. In particular, there is a height 1 non-crossing partition u such that $t = g_u$, and

$c' = g_{\overline{u}}$. By Proposition 4.2, only reflections in the parabolic subgroup G' fixing $\ker(c'-1)$ may appear in reduced T-decompositions of c'; let T' be the set of reflections in G'. We discuss by cases:

- If t' is (short) symmetric, $\overline{u}$ is long symmetric. Denote by $\overline{u}_1$ the long symmetric element of $NCP(e,e,n+1)$ with only non-trivial part this long part. Denote by $\overline{u}_2$ the (short symmetric) element of $NCP(e,e,n+1)$ with non-singleton parts the remaining non-singleton parts of $\overline{u}$. The elements $c'_1 := g_{\overline{u}_1}$ and $c'_2 := g_{\overline{u}_2}$ satisfy $c'_1 c'_2 = c'_2 c'_1 = c'$. They are Coxeter

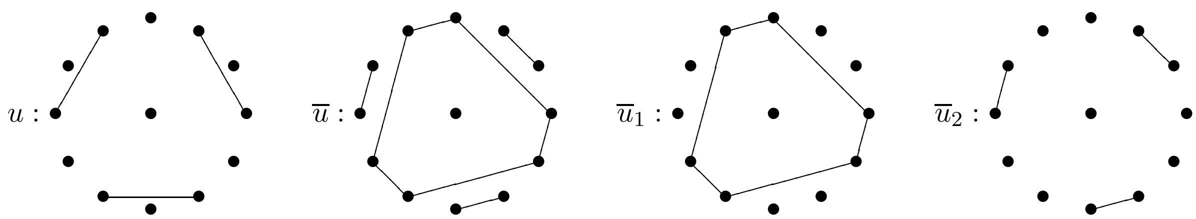

FIGURE 12. Decomposing the partition $\overline{u}$, when u is symmetric

  elements in parabolic subgroups G'_1 and G'_2 of respective types $G(e,e,p)$ and $G(1,1,q)$, where p and q satisfy $p+q=n$. Set $A'_1 := \{t \in T | t \preccurlyeq_T c'_1\}$, $A'_2 := \{t \in T | t \preccurlyeq_T c'_2\}$ and $A' := \{t \in T | t \preccurlyeq_T c'\}$. Elements of A'_1 (resp. A'_2) correspond to height 1 non-crossing partitions below $\overline{u}_1$ (resp. $\overline{u}_2$), and $A' = \pi(A) \cap W' = A'_1 \cup A'_2$. Since $g' \preccurlyeq c'$, a reduced T-decomposition of g' consists of elements of A'. The elements of A'_1 commute with the elements of A'_2; by regrouping them, we obtain a decomposition $g' = g'_1 g'_2$, with $g'_1 \in G'_1$ and $g'_2 \in G'_2$ satisfying $g'_1 \preccurlyeq_{T'_1} c'_1$ and $g'_2 \preccurlyeq_{T'_2} c'_2$, where $T'_1 = T \cap G'_1$ and $T'_2 = T \cap G'_2$. By induction assumption, $g'_1 = g_v$, where v belongs to $NCP(e,e,p)$ (or, more rigorously, to the image of $NCP(e,e,p)$ under the identification of $NCP(e,e,p)$ with elements of $NCP(e,e,n+1)$ below $\overline{u}_1$). Using the similar result (from [B1]) for the type A case, we also know that $g'_2 = g_w$, where w belongs to $NCP(1,1,q)$ (or, more rigorously, to the image of $NCP(1,1,q)$ under the identification of $NCP(1,1,q)$ with elements of $NCP(e,e,n+1)$ below $\overline{u}_2$; the identification sends each non-singleton part to e copies, one for each element of the orbit for the action of μ_e on the set of parts of $\overline{u}_2$). Set $u' := v \vee w$. We have $g' = g_{u'}$. The partition $u \vee u'$ is easy to construct, and one may observe that $u \backslash (u \vee u') = u'$. This implies that $g_{u \vee u'} = g_u g_{u'} = tg' = g$. We have proved our claim that $g \in P_G$.
- When u is asymmetric, the parabolic subgroup G' is of type $G(1,1,n+1)$. We conclude with

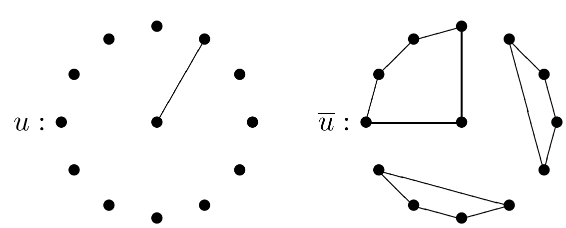

FIGURE 13. The element $\overline{u}$ when u is asymmetric.

  a discussion similar (but simpler) to the above one (the identification of $NCP(1,1,n+1)$ with the interval $[1,\overline{u}]$ in $NCP(e,e,n+1)$ is Lemma 1.19 (ii)).

□

Lemma 4.5. *Let $g \in P_G$. Choose $(t_1, \dots, t_k) \in \mathrm{Red}_T(g)$. Consider the non-crossing partitions $u_1, \dots, u_k$ of height 1 corresponding to $t_1, \dots, t_k$. Then $g_{u_1 \vee \cdots \vee u_k} = g$. The element $u_1 \vee \cdots \vee u_k$ only depends on g, and not on the choice of $(t_1, \dots, t_k)$ in $Red_T(g)$.*

Proof. By assumption, there is an element $w \in NCP(e, e, n+1)$ such that $g_w = g$. By 4.1 (ii), we have

$$\mathrm{codim}(\ker(g-1)) = l_T(g) = k.$$

Since $g = t_1 \dots t_k$, the only possibility is that the reflecting hyperplanes of $t_1 \dots t_k$ intersect transversally and $\ker(g-1) = \bigcap_{i=1}^k \ker(t_i - 1)$.

Set $u := u_1 \vee \cdots \vee u_k$. By 4.1 (iii), we have $\ker(g_u - 1) \subseteq \bigcap_{i=1}^k \ker(t_i - 1) = \ker(g_w - 1)$ thus, by Proposition 4.2, $w \preccurlyeq u$. On the other hand, for all i, $\ker(g_w - 1) \subseteq \ker(t_i - 1) = \ker(g_{u_i} - 1)$, thus $u_i \preccurlyeq w$. We conclude, using the definition of the "meet" operation, that $u = w$. □

With the notations of the lemma, set $u_g := u_1 \vee \cdots \vee u_k$. The lemma precisely asserts that $g \mapsto u_g$ is an inverse of the map $NCP(e, e, n+1) \to G(e, e, n+1), u \mapsto g_u$. It is then clear that we have the following theorem, which summarises the results of this section:

Theorem 4.6. *The diagram*

$$\begin{tikzcd}
 & P_B \arrow[dd, "\pi"] \\
NCP(e,e,n+1) \arrow[ur, "u \mapsto b_u"] & \\
 & P_G \arrow[ul, "g \mapsto u_g"]
\end{tikzcd}$$

is commutative. Its arrows are poset isomorphisms, where

- *$NCP(e, e, n+1)$ is endowed with the "is finer than" relation $\preccurlyeq$,*
- *P_B is endowed with the relation $\preccurlyeq_{P_B}$ defined by $\forall b, b' \in P_B, b' \preccurlyeq_{P_B} b \Leftrightarrow \exists b'' \in P_B, b'b'' = b$,*
- *P_G is endowed with the "is initial segment of reduced T-decomposition" relation $\preccurlyeq_T$.*

In the above theorem, the composed map $g \mapsto b_{u_g}$ is an analog of the "Tits section" from a finite Coxeter group to its Artin group.

5. A Garside structure for $B(e, e, n+1)$

Definition 5.1. *The* dual braid monoid *of type $(e, e, n+1)$ is the submonoid $M(e, e, n+1)$ of $B(e, e, n+1)$ generated by P_B.*

We prove in this section that $M(e, e, n+1)$ is Garside monoid, with Garside element b_{triv} and set of simples P_B. The proof uses the results from the previous section, about reduced decompositions in $G(e, e, n+1)$, and is very similar to the proof in [BDM].

Let us complete the commutative diagram of Theorem 4.6 by adding the natural inclusions:

$$\begin{tikzcd}
& P_B \arrow[r, hook] \arrow[dd, "\pi"] & M(e,e,n+1) \arrow[r, hook] & B(e,e,n+1) \arrow[dd, two heads, "\pi"] \\
NCP(e,e,n+1) \arrow[ur] & & & \\
& P_G \arrow[ul] \arrow[rr, hook] & & G(e,e,n+1)
\end{tikzcd}$$

Theorem 5.2. *The monoid $M(e, e, n+1)$ is a Garside monoid. The element b_{triv} is a Garside element, for which the set of simple elements is P_B. The atoms of $M(e, e, n+1)$ are the images by α of height 1 non-crossing partitions.*

The inclusion $M(e, e, n+1) \hookrightarrow B(e, e, n+1)$ identifies $B(e, e, n+1)$ with the group of fractions of $M(e, e, n+1)$. Hence $B(e, e, n+1)$ is a Garside group.

For general properties of Garside groups, a good reference is [Dh1].

Proof. Since $P_G = \{g \in G(e, e, n+1) | g \preccurlyeq_T c\}$ (Proposition 4.4), it may be endowed with the partial product structure defined in section 0.4 of [B1] (note that, since T is stable under conjugacy, the relations $\prec_T$ and $\succ_T$ of *loc. cit.* coincide – in particular, c is balanced). By Theorem 4.6, $(P_G, \preccurlyeq_T)$ is isomorphic to $(NCP(e, e, n+1), \preccurlyeq)$, and in particular (Lemma 1.17) it is a lattice. This implies, using [B1, Theorem 0.5.2], that the monoid $\mathbf{M}(P_G)$ generated by P_G is a Garside monoid (with Garside element corresponding to c, and set of simples corresponding to P_G).

The bijection $P_G \simeq P_B$ of Theorem 4.6 is compatible with the partial product, it induces a monoid morphism $\phi : \mathbf{M}(P_G) \to M(e, e, n+1)$, which extends to a group morphism $\psi : \mathbf{G}(P_G) \to G(e, e, n+1)$. We have $\phi(c) = b_{\mathrm{triv}}$. The point is to prove that ϕ and ψ are isomorphisms. By definition, P_B generates $M(e, e, n+1)$, thus ϕ is surjective. By Theorem 3.6, P_B contains group generators for $B(e, e, n+1)$, so ψ is also surjective. Since $\mathbf{M}(P_G)$ is a Garside monoid, the canonical map $\mathbf{M}(P_G) \to \mathbf{G}(P_G)$ is injective. Thus, to complete the proof, it is sufficient to prove that ψ is injective.

To prove the injectivity of ψ, it is sufficient to check that the Broué-Malle-Rouquier relations between the specific generators in Theorem 3.6 are consequences of the relations in $\mathbf{G}(P_G)$. We illustrate this straightforward computation with the relation $\langle \tau_2 \tau_2' \rangle^e = \langle \tau_2' \tau_2 \rangle^e$. The generator τ_2 (resp. τ_2') is identified with a_0 (resp. a_n). The least upper bound of the partitions u_0 and u_n is the height 2 partition $v := ((u_0)^\sharp)_* = ((u_n)^\sharp)_*$, whose only non-singleton part is $\mu_e \cup \{0\}$. By definition of the complement operation, we have $u_0 \backslash v = u_n$. By Proposition 2.4, we then have $a_0 a_n = b_v$. This relation holds in P_B, thus is P_G (via the isomorphism of Theorem 4.6) and in $\mathbf{G}(P_G)$. By symmetry, we have relations $a_{kn} a_{(k+1)n} = b_v$, for all integer k. If e is even, we have

$$\langle a_0^{(0)} a_0^{(1)} \rangle^e = (a_0 a_n)^{e/2} = b_v^{e/2} = a_n a_{2n} a_{3n} a_{4n} \dots a_{(e-1)n} a_{en} = a_n b_v^{e/2-1} a_0 = \langle a_n a_0 \rangle^e$$

as a consequence of relations of $\mathbf{G}(P_G)$. The e odd case, as well as the other relations of Theorem 3.5, may be obtained in a similar manner. □

An application to centralisers of periodic elements. The main result in [BDM] is a description of centralisers of roots of central elements in type A braid groups. Similarly, we obtain the following result:

Proposition 5.3. *The element* $b_{\mathrm{triv}}^{\overline{\frac{en}{e\wedge(n+1)}}}$ *generates the center of* $B(e,e,n+1)$. *Let* $k \in \mathbb{Z}_{\geq 1}$, *and let* $k' := \frac{en}{k(n+1)\wedge en}$. *Assume* $k' \neq 1$. *Then the centraliser* $C_{B(e,e,n+1)}(b_{\mathrm{triv}}^k)$ *is isomorphic to* $B(k',1,en/k')$.

Proof. The first statement, about $b_{\mathrm{triv}}^{\overline{\frac{en}{e\wedge(n+1)}}}$, may already be found in [BMR].

Let $k \in \mathbb{Z}$.

Let $b \in P_B$. There is a unique $u \in NCP(e,e,n+1)$ such that $b = b_u$. Since $b_u b_{\overline{u}} = b_{\mathrm{triv}} = b_{\overline{u}} b_{\overline{\overline{u}}}$, we have

$$b_{\overline{\overline{u}}} = b_{\mathrm{triv}}^{-1} b_u b_{\mathrm{triv}}.$$

Replacing $\overline{\overline{u}}$ by its description from Lemma 1.22 (v), we obtain $b_{\zeta_{en}^{n+1}u} = b_{\mathrm{triv}}^{-1} b_u b_{\mathrm{triv}}$ and

$$b_{\zeta_{en}^{k(n+1)}u} = b_{\mathrm{triv}}^{-k} b_u b_{\mathrm{triv}}^k. \tag{1}$$

Let $b \in C_{B(e,e,n+1)}(b_{\mathrm{triv}}^k)$. Write b in Garside normal form: $b = b_{\mathrm{triv}}^N b_1 \dots b_m$, with $N \in \mathbb{Z}$ and $b_i \in P_B$. By (1), conjugating by a power of b_{triv} preserves P_B. If b is invariant by such a conjugacy, the unicity of the normal form implies that each b_i is invariant by the conjugacy, that is, using (1), we have $\zeta_{en}^{k(n+1)} u_i = u_i$, where u_i is the non-crossing partition such that $b_i = b_{u_i}$.

The integer k' is the order of $\zeta' := \zeta_{en}^{k(n+1)}$. If $k' \neq 1$, multiplication by ζ' fixes no asymmetric partition. An symmetric partition is fixed if and only if it lies in $NCP(k',1,en/k')$. This implies that the natural morphism $B(k',1,en/k') \to C_{B(e,e,n+1)}(b_{\mathrm{triv}}^k)$ is surjective.

To prove the injectivity, one proceeds as follows. By [BDM], Proposition 3.26, the monoid $C_{M(e,e,n+1)}(b_{\mathrm{triv}}^k)$ is a Garside monoid, with set of simples in bijection with $NCP(k',1,en/k')$. There is a "dual monoid" $M(k',1,en/k')$ for $B(k',1,en/k')$, with $NCP(k',1,en/k')$. The bijection between their set of simples induces an isomorphism $M(k',1,en/k') \simeq C_{M(e,e,n+1)}(b_{\mathrm{triv}}^k)$. ☐

6. Transitivity of Hurwitz action

This section contains complements about the structure of $\mathrm{Red}_T(c)$, which will be needed in the next section to write a simple presentation for $G(e,e,n+1)$.

Let $g \in G(e,e,n+1)$, with $l_T(g) = k$. For any $i = 1,\dots,k-1$ and any $(t_1,\dots,t_k) \in \mathrm{Red}_T(g)$, set

$$\sigma_i((t_1,\dots,t_k)) := (t_1,\dots,t_{i-1}, t_i t_{i+1} t_i^{-1}, t_i, t_{i+2},\dots,t_k).$$

The right-hand side clearly belongs to $\mathrm{Red}_T(g)$, and one checks that $\sigma_1,\dots,\sigma_{k-1}$ satisfy the defining the classical braid relations defining the Artin group of type A_{k-1}. The corresponding action of this Artin group on $\mathrm{Red}_T(g)$ is called *Hurwitz action.* The following result extends [B1], Proposition 1.6.1.

Proposition 6.1. *Let* $g \in P_G$. *The Hurwitz action is transitive on* $\mathrm{Red}_T(g)$.

We need two lemmas:

Lemma 6.2. *Let* w *be a maximal short symmetric element of* $NCP(e,e,n+1)$. *We have* $ht(w) = n-1 = l_T(g_w)$. *There exists an element* $(t_1,\dots,t_{n-1}) \in \mathrm{Red}_T(g_w)$ *such that for all symmetric* $t \in T \cap P_G$, *one may find* $i \in \{1,\dots,n-1\}$ *and* $\zeta \in \mu_{en}$ *such that* $t = \zeta t_i$.

Proof. The statement about $ht(w)$ is easy. Multiplication by μ_{en} is transitive on the set of maximal short symmetric elements, so it suffices to prove the lemma for a particular one.

Consider the short symmetric non-crossing partition u whose non-singleton parts are

$$\{1,\zeta_{en}\},\{\zeta_{en}^{-1},\zeta_{en}^{2}\},\ldots,\{\zeta_{en}^{-i+1},\zeta_{en}^{i}\},\ldots,\{\zeta_{en}^{-\lceil\frac{n-1}{2}\rceil+1},\zeta_{en}^{\lceil\frac{n-1}{2}\rceil}\}$$

and their images under multiplication by μ_e; consider the short symmetric non-crossing partition v whose non-singleton parts are

$$\{1,\zeta_{en}^{2}\},\{\zeta_{en}^{-1},\zeta_{en}^{3}\},\ldots,\{\zeta_{en}^{-i+1},\zeta_{en}^{i+1}\},\ldots,\{\zeta_{en}^{-\lfloor\frac{n-1}{2}\rfloor+1},\zeta_{en}^{\lfloor\frac{n-1}{2}\rfloor+1}\}$$

and their images under multiplication by μ_e. For all integer $p\geq 0$, set $s_p:=t_{-p,p+1}$ and $s'_p:=t_{2p+1,-2p-1}$. The element $u\vee v$ is a maximal short symmetric non-crossing partition, with parts

$$\{0\},\{\zeta_{en}^{-\lceil\frac{n-1}{2}\rceil+1},\zeta_{en}^{-\lceil\frac{n-1}{2}\rceil+2},\ldots,\zeta_{en}^{\lfloor\frac{n-1}{2}\rfloor+1}\}$$

and their images under multiplication by μ_e.

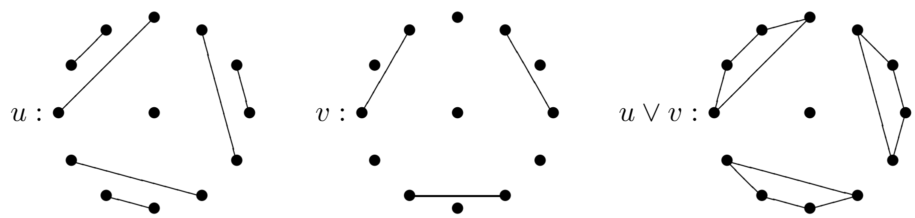

FIGURE 14. Illustration in $NCP(3,3,5)$

Set $U:=\{w\in NCP(e,e,n+1)|w\preccurlyeq u, ht(w)=1\}$ and $V:=\{w\in NCP(e,e,n+1)|w\preccurlyeq v, ht(w)=1\}$ The lemma follows from the following simple observations:

- $ht(u)=\lceil(n-1)/2\rceil=|U|$ and $ht(v)=\lfloor(n-1)/2\rfloor=|V|$;
- any reduced decomposition of g_u (resp. g_v) involves all the reflections associated to elements of U (resp. V);
- $v=u\backslash(u\vee v)$, thus $g_ug_v=g_{u\vee v}$;
- $u\vee v$ is a maximal short symmetric element which admits a reduced T-decomposition involving all elements in $U\cup V$;
- any height 1 symmetric non-crossing partitions is obtained by multiplying some element of $U\cup V$ by some element in μ_{en}.

□

Lemma 6.3. *Let w' be a minimal long symmetric element of $NCP(e,e,n+1)$. We have $ht(w')=2=l_T(g'_w)$. Let u_p be a height 1 asymmetric element finer than w'. For all integers k, $u_{p+kn}\preccurlyeq w'$, and*

$$\mathrm{Red}_T(g_{w'})=\{(g_{u_p},g_{u_{p+n}}),(g_{u_{p+n}},g_{u_{p+2n}}),\ldots,(g_{u_{p+(e-1)n}},g_{u_p})\}.$$

Proof. The partition w' has only one non-singleton part, of the form $\{0,\zeta,\zeta_e\zeta,\zeta_e^2\zeta,\ldots,\zeta_e^{e-1}\zeta\}$. One observes that height 1 elements finer than w' are the asymmetric partitions v_i with non-singleton

part $\{0, \zeta_e^i \zeta\}$. By Proposition 4.2 (i) $\Leftrightarrow$ (ii), all sequences in $\mathrm{Red}_T(w')$ have their terms among the g_{v_i}'s. One has $v_i \backslash w' = v_{i+1}$. The result follows. $\square$

Proof of the proposition. We prove the proposition by induction. Assume it to be known for all values of n smaller than the considered one.

Consider first the particular case when $g = c$. Choose w maximal short symmetric in $NCP(e,e,n+1)$. Set $w' := w \backslash \mathrm{triv}$. The element w' is minimal long symmetric. Choose $(t_0, \dots, t_{n-2}) \in \mathrm{Red}_T(w)$ satisfying the condition of Lemma 6.2. Let $(t_{n-1}, t_n) \in \mathrm{Red}_T(w')$. We have $(t_0, \dots, t_n) \in \mathrm{Red}_T(c)$. Let $(t'_0, \dots, t'_n)$ be another element of $\mathrm{Red}_T(c)$.

Claim. *We may find $(t''_0, \dots, t''_n)$ in the Hurwitz orbit of $(t_0, \dots, t_n)$ such that $t''_0 = t'_0$.*

Proof of the claim. One first remarks that it suffices to find $(t''_0, \dots, t''_n)$ with $t''_i = t'_0$, since one may use Hurwitz to "slide" t''_i to the beginning of the sequence. A standard calculation shows that the action of $\beta := (\sigma_1 \dots \sigma_n)^{n+1}$ on $\mathrm{Red}_T(c)$ coincides with term-by-term conjugacy by c, which, on the corresponding partitions, coincides with $u \mapsto \overline{\overline{u}}$, which itself coincides with the multiplication by ζ_{en}^{n+1} (Lemma 1.22 (v)). By Proposition 4.2, $t'_0 \in P_G$. If t'_0 is associated to a symmetric height 1 partition, then, using Lemma 6.2, we may apply a suitable power of β to find in the Hurwitz orbit of $(t_0, \dots, t_n)$ a sequence involving t'_0 (on symmetric partitions, multiplying by ζ_{en}^{n+1} is the same as multiplying by ζ_{en}). It t'_0 is asymmetric, the underlying partition is obtained from the underlying partition of t_{n-1} by multiplying by some $\zeta \in \mu_{en}$. Use σ_n to multiply by ζ_e the underlying partitions of t_{n-1} and t_n (Lemma 6.3); use β to multiply by ζ_{en}^{n+1}; to conclude, observe that ζ_e and $\zeta_{en}^{n+1} = \zeta_e \zeta_{en}$ generate μ_{en}.

To conclude that $(t_0, \dots, t_n)$ and $(t'_0, \dots, t'_n)$ lie in the same Hurwitz orbit, it is enough to prove that $(t'_1, \dots, t'_n)$ and $(t''_1, \dots, t''_n)$ lie in the same Hurwitz orbit. This follows either from the induction assumption (when the parabolic subgroup fixing $\ker(t'_0 c - 1)$ is of type $G(e,e,n)$) or from [B1] 1.6.1 (when this parabolic subgroup is a real reflection group).

Similarly, the general case follows either from the induction assumption or from [B1] 1.6.1, depending on the type of the parabolic subgroup W' fixing $\ker(g-1)$. This is because $\mathrm{Red}_T(g)$ coincides with $\mathrm{Red}_{T'}(g)$, where $T' = T \cap W'$ (Proposition 4.2). $\square$

7. Explicit presentations of $M(e,e,n+1)$ and $B(e,e,n+1)$

The generating set, in the theorem below, is the set A of elements associated with non-crossing partitions of height 1 (section 3). By Theorem 4.6, it is in bijection with $T \cap P_c$. Alternatively, we have $A = \{a \in P_B | l(a) = 1\}$ (where l is the canonical length function on $B(e,e,n+1)$). The elements of A were explicitely described in section 3: there are $n(n-1)$ symmetric elements $a_{p,q}$, and en asymmetric elements a_r (where p,q,r are arbitrary integers such that $|p-q| < n$; recall that $a_{p,q} = a_{q,p} = a_{p+n,q+n}$ and $a_p = a_{p+en}$).

Theorem 7.1. *(1) The following length 2 relations between elements of A hold in $M(e,e,n+1)$:*
For every quadruple (p,q,r,s) of integers such that $p < q < r < s < p+n$, there are relations:

$$
\begin{array}{ll}
(\mathcal{R}_1) & a_{p,q} a_{r,s} = a_{r,s} a_{p,q} \\
(\mathcal{R}_2) & a_{p,s} a_{q,r} = a_{q,r} a_{p,s}.
\end{array}
$$

For every triple (p,q,r) of integers such that $p<q<r<p+n$, there are relations:

$$(\mathcal{R}_3) \quad a_{p,r}a_{q,r} = a_{q,r}a_{p,q} = a_{p,q}a_{p,r}$$
$$(\mathcal{R}_4) \quad a_{p,q}a_r = a_r a_{p,q}.$$

For every pair (p,q) of integers such that $p<q<p+n$, there are relations:

$$(\mathcal{R}_5) \quad a_{p,q}a_p = a_p a_q = a_q a_{p,q}$$

For every $p \in \mathbb{Z}/n\mathbb{Z}$, there are relations:

$$(\mathcal{R}_6) \qquad a_p a_{p+n} = \cdots = a_{p+in}a_{p+(i+1)n} = \cdots = a_{p+(e-1)n}a_p.$$

(2) Denote by $\mathcal{R}$ the set of all relations considered in (1). We have presentations:

$$M(e,e,n+1) \simeq \langle A \,|\, \mathcal{R} \rangle_{\text{Monoid}} \qquad \textit{and} \qquad B(e,e,n+1) \simeq \langle A \,|\, \mathcal{R} \rangle_{\text{Group}}.$$

On the graphical illustration (Figure 15), one may realise that: generators commute when the non-trivial parts ("edges") of their associated partitions do not interest; they satisfy another length 2 relation when the edges have common endpoints; there is no relation when edges cross in their inner part. Note also than, whenever u,v,w are such that $ht(u)=1$, $ht(v)=1$, $ht(w)=2$, $u \preccurlyeq w$ and $v = u\backslash w$, we have a relation $b_u b_v = b_v b_{v\backslash w}$, and that all relations are of this form. Relations are in correspondence with non-crossing partitions of height 2. This allows a more intrinsic (though less explicit) reformulation of the theorem.

Proof. The observation that the relations are of the form $b_u b_v = b_v b_{v\backslash w}$ suffices to prove (1).

To prove (2), using standard facts about Garside structures (see [B1], Section 0), we deduce from Theorem 5.2 a monoid presentation for $M(e,e,n+1)$ with:

- Generating set: P_G.
- Relations: $g'g'' = g$ whenever the relation holds in $G(e,e,n+1)$ and $l_T(g') + l_T(g'') = l_T(g)$.

One rewrites this presentation as follows:

- Generating set: $P_G \cap T$.
- Relations: $t_1 \dots t_k = t'_1 \dots t'_k$ whenever the relation holds in G and $(t_1,\dots,t_k) \in \mathrm{Red}_T(t_1 \dots t_k)$ and $(t'_1,\dots,t'_k) \in \mathrm{Red}_T(t'_1 \dots t'_k) = \mathrm{Red}_T(t_1 \dots t_k)$.

Now Proposition 6.1 implies the above relations, for $k \geq 1$, are consequences of the relations for $k=2$. When considering the presentation as a group presentation, we obtain a presentation for the group of fractions $G(e,e,n+1)$ of $M(e,e,n+1)$. □

The presentation has some symmetries:

Proposition 7.2. *The monoid $M(e,e,n+1)$ is isomorphic to the opposed monoid $M(e,e,n+1)^{op}$.*

Proof. Consider an plane axial symmetry ϕ preserving the regular ne-gon μ_{ne}. By its action on diagrams, ϕ induces an involution of the generating set A. The relations of Proposition 7.1 only depend on (oriented) incidence of diagrams. By examining their graphical interpretations, one may observe that if $ab = cd$ is a relation in $\mathcal{R}$ between $a,b,c,d \in A$, then $\phi(b)\phi(a) = \phi(d)\phi(c)$ is also in $\mathcal{R}$. Hence ϕ realizes an anti-automorphism of $M(e,e,n+1)$. □

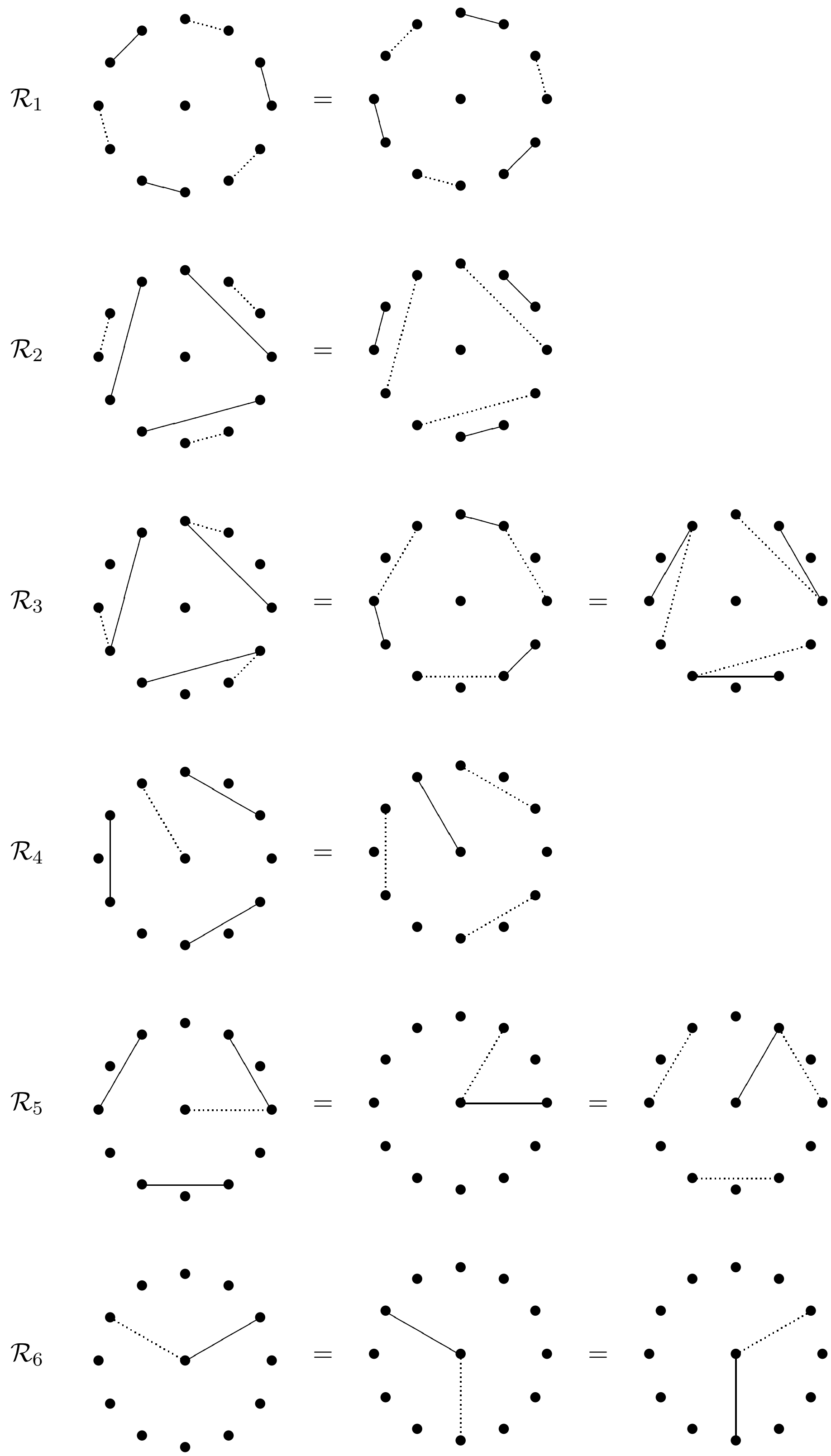

FIGURE 15. Diagrammatic interpretation of the relations in $G(3,3,5)$. All relations involves products of two generators. The full segments represent the first term, the dotted segments represent the second term.

Note that the above proof does not provide us with a natural antiautomorphism of $M(e, e, n+1)$. By combining all axial symmetries and rotations of μ_{ne}, we obtain a dihedral group of order $2ne$, whose even (resp. odd) elements are automorphisms (resp. antiautomorphisms) of $M(e, e, n+1)$. The situation is similar to the one in [B1], Section 2.4.

Remark. Our presentation also makes it easy to identify $B(e, 1, n)$ with a subgroup of $B(e, e, n+1)$. For all $p \in \mathbb{Z}/n\mathbb{Z}$, set $b_p := a_p a_{p+n}$ (by $\mathcal{R}_6$, we have $b_p := a_{p+in} a_{p+(i+1)n}$ for all integer i). Then the subgroup generated by the $a_{p,q}$'s and the b_p's is isomorphic to $B(e, 1, n)$. The sublattice $NCP(e, 1, n) \hookrightarrow NCP(e, e, n+1)$ is the lattice of simple elements of a Garside structure for $B(e, 1, n)$, whose atoms are the $a_{p,q}$'s and the b_p's. These atoms generate a monoid $M(e, 1, n)$. Actually, up to isomorphism, the structures of $NCP(e, 1, n)$, $B(e, 1, n)$ and $M(e, 1, n)$ do not depend on $e \geq 2$. When $e = 2$, we recover the dual braid monoid of type B_n.

8. Zeta polynomials, reflection degrees and Catalan numbers

This section contains some complements about the combinatorics of the lattice $NCP(e, e, n+1)$, which is the lattice of simple elements in our Garside structure for $B(e, e, n+1)$.

By a *chain of length* N in a poset $(P, \leq)$, we mean a finite weakly increasing sequence $p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_N$ of elements of P. When the poset is finite, one may consider the *Zeta function* Z_P, whose value at a positive integer N is the number of chains of length $N-1$ in P. For example,

$$Z_P(1) = 1, \ Z_P(2) = |P|.$$

Let W be an irreducible real reflection group of rank r, with reflection degrees $d_1 \leq d_2 \leq \cdots \leq d_r$ (these are the degrees of homogeneous algebraically independent generators of the algebra of W-invariant polynomial functions; d_r, often denoted by h, is the Coxeter number). Let $P(W)$ be the lattice of simple elements in the dual braid monoid attached to W. The following general formula was suggested by Chapoton, and has now been proved using the work of Reiner and Athanasiadis-Reiner for the classical types, and computer checks by Reiner for the exceptional types ([Ch], [R], [AR]):

$$Z_{P(W)}(X) = \prod_{i=1}^{n} \frac{d_i + d_n(X-1)}{d_i}.$$

In particular, one has

$$|P(W)| = \prod_{i=1}^{n} \frac{d_i + d_n}{d_i}.$$

Inspired by the type A situation, the number $\prod_{i=1}^{n} \frac{d_i+d_n}{d_i}$ is called *Catalan number* attached to W.

It is natural to expect Chapoton's formula to continue to hold for our "dual monoid" of type $G(e, e, n+1)$.

We set

$$Z(1, 1, n)(X) := \prod_{k=2}^{n} \frac{k + n(X-1)}{k},$$

$$Z(e, 1, n)(X) := \prod_{k=1}^{n} \frac{ek + en(X-1)}{ek} = \prod_{k=1}^{n} \frac{k + n(X-1)}{k}$$

and

$$Z(e,e,n+1)(X) := \frac{n+1+en(X-1)}{n+1}\prod_{k=1}^{n}\frac{k+n(X-1)}{k}.$$

Since $2,3,\ldots,n-1,n$ (resp. $e,2e,\ldots,e(n-1),en$, resp. $e,2e,\ldots,e(n-1),en,n+1$) are the reflection degrees of $G(1,1,n)$ in its irreducible reflection representation (resp. $G(e,1,n)$, resp. $G(e,e,n+1)$), these terms are the right-hand sides in the corresponding Chapoton's formulae.

One observes the relations

$$Z(e,e,n+1) = (1+\frac{en}{n+1}(X-1))Z(e,1,n)$$

and

$$Z(e,1,n) = (1+n(X-1))Z(1,1,n).$$

Theorem 8.1. *For any $e \geq 2$, $n \geq 1$, we have $Z_{NCP(1,1,n)} = Z(1,1,n)$, $Z_{NCP(e,1,n)} = Z(e,1,n)$ and $Z_{NCP(e,e,n+1)} = Z(e,e,n+1)$.*

Corollary 8.2. *The cardinal of $NCP(e,1,n)$, resp. $NCP(e,e,n+1)$, is the "Catalan number" attached to $G(e,1,n)$, resp. $G(e,e,n+1)$.*

Another useful consequence of the theorem is that the cardinal of $\mathrm{Red}_T(c)$ may be computed, since it coincides with the number of strict $n+1$ chains (knowing the number of weak k-chains for $k=1,\ldots,N$, one obtains the number of strict N-chains using a straightforward inversion formula).

Since $G(1,1,n)$, $G(2,1,n)$, $G(2,2,n+1)$ and $G(e,e,2)$ are real, the corresponding cases are already known (see [Ch]; the $G(2,2,n+1) = W(D_{n+1})$ case was first proved by Athanasiadis-Reiner, [AR]). Note that $Z(e,1,n)$ does not depend on $e \geq 2$ – the poset $NCP(e,1,n)$ is actually isomorphic to $NCP(2,1,n)$, but is worth considering due to its natural relationships with $NCP(e,e,n+1)$, which does depend on e.

The rest of the section is devoted to the proof of the theorem. The general strategy is to prove bijectively, at the level of chains, the relations $Z_{NCP(e,1,n)} = (1+n(X-1))Z_{NCP(1,1,n)}$ and $Z_{NCP(e,e,n+1)} = (1+\frac{n+1}{en}(X-1))Z_{NCP(e,1,n)}$.

Consider the poset morphism $E : NCP(e,1,n) \to NCP(1,1,n)$ defined as follows: for any u in $NCP(e,1,n)$, $E(u)$ is the element of $NCP(1,1,n)$ whose parts are the images of the parts of u by the map $z \mapsto z^e$ (one checks that this indeed makes sense). This map E naturally generalises the map $\mathrm{Abs} : NCP(2,1,n) \to NCP(1,1,n)$ constructed by Biane-Goodman-Nica ([BGN]).

For all $\zeta \in \mu_{en}$, we denote by s_ζ the short element of $NCP(e,1,n)$ whose parts are $\{0\}$, $\{\zeta, \zeta_{en}\zeta, \ldots, \zeta_{en}^{n-1}\zeta\}$ and their images under the μ_e-action. The element s_ζ is maximal among short symmetric elements. There are n such elements (we have $s_\zeta = s_{\zeta_e\zeta}$). The element $l_\zeta := \overline{s_\zeta}$ is a minimal long asymmetric element.

Lemma 8.3. (i) *The map E is a poset morphism. For all $u \preccurlyeq v$ in $NCP(e,1,n)$, one has $E(u\backslash v) = E(u)\backslash E(v)$. In particular, $\overline{E(u)} = E(\overline{u})$.*

(ii) *Let $\zeta \in \mu_{en}$. The map E restricts to a poset isomorphism from $[\mathrm{disc}, s_\zeta]$ to $NCP(1,1,n)$.*

(iii) *Let $\zeta \in \mu_{en}$. The map E restricts to a poset isomorphism from $[l_\zeta, \mathrm{triv}]$ to $NCP(1,1,n)$.*

Proof. (i) is easy.

(ii). Consider the restriction map res from $[\mathrm{disc}, s_\zeta]$ to $NCP_{\{\zeta,\zeta_{en}\zeta,\ldots,\zeta_{en}^{n-1}\zeta\}}$. Clearly, res is an isomorphism, and the natural map $NCP_{\{\zeta,\zeta_{en}\zeta,\ldots,\zeta_{en}^{n-1}\zeta\}} \to NCP(1,1,n)$ is an isomorphism. By composing them, one obtains the desired isomorphism from $[\mathrm{disc}, s_\zeta]$ to $NCP(1,1,n)$.

(iii) follows from (i) + (ii). $\square$

One could define a category of *complemented lattices*, axiomatising the properties of our lattices and of the operations $\backslash$. The statement (i) in the above lemma would then express that E is a *morphism of complemented lattices*. Similarly, the case (A) of the definition of the complement operation in $NCP(e, e, n + 1)$ is designed to make $*$ a morphism of complemented lattices. The definition below could also be generalised to study chains in any complemented lattices. This axiomatic approach is not necessary to the present work – the systematic investigation could however be interesting.

Definition 8.4. *Let P be one of the posets $NCP(1, 1, n)$, $NCP(e, 1, n)$ or $NCP(e, e, n + 1)$, endowed with its operations $\vee$, $\backslash$, etc... Let $N \in \mathbb{Z}_{\geq 0}$. An $(N + 1)$*-derived sequence *in P is a sequence $p = (p_0, \ldots, p_N)$ of elements of P such that, for all i,*

$$p_i = (p_1 \vee \cdots \vee p_{i-1})\backslash(p_1 \vee \cdots \vee p_i)$$

and $p_1 \vee \cdots \vee p_N = \mathrm{triv}$.

To any N-chain $u = (u_1, \ldots, u_N)$, we associate an $(N + 1)$-derived sequence ∂u, defined by

$$\partial u := (\mathrm{disc}\backslash u_1 = u_1, u_1\backslash u_2, u_2\backslash u_3, \ldots, u_{N-1}\backslash u_N, u_N\backslash \mathrm{triv} = \overline{u_N}).$$

To any $(N + 1)$-derived sequence $p = (p_0, \ldots, p_N)$, we associate an N-chain $\int p$, defined by

$$\int p := (p_0, p_0 \vee p_1, p_0 \vee p_1 \vee p_2, \ldots, p_0 \vee p_1 \vee \cdots \vee p_{N-1}).$$

Checking that ∂u is indeed a derived sequence, and that $\int p$ is indeed a chain, is trivial. It is also trivial to check that

Lemma 8.5. *The maps ∂ and $\int$ are reciprocal bijections between the sets of N-chains and the set of $(N+1)$-derived sequences in the corresponding lattice. We have $\partial E = E\partial$, $\int E = E \int$, $\partial * = *\partial$, $\int * = * \int$.*

where we still denote by E the operation sending a sequence $(s_1, \ldots, s_k)$ to $(E(s_1), \ldots, E(s_k))$, and similarly for $*$.

The first section of [BGN] inspired both the following lemma and its proof.

Lemma 8.6. *For any e, n, N, the map E from $(N + 1)$-derived sequences in $NCP(e, 1, n)$ to $(N + 1)$-derived sequences in $NCP(1, 1, n)$ is $(1 + nN)$-to-1 fibration (*i.e.*, the pre-image of any $(N + 1)$-derived sequence in $NCP(1, 1, n)$ has cardinal $1 + nN$).*

Proof. Observe first that any derived sequence $p = (p_0, \ldots, p_N)$ in $NCP(e, 1, n)$ contains exactly one long element: let i be the first integer such $p_0 \vee \cdots \vee p_i$ is long; then p_i, being the complement of a short element in a long element, is long. All other elements are either complements of short elements in short elements, or complements of long elements in long elements – in both cases, they must be short (Lemma 1.14).

Let $q := E(p)$. For all j, let m_j be the number of parts of q_j. We have

$$\begin{aligned}\sum_{j=0}^{N} m_j = \sum_{j=0}^{N}(n - ht(q_j)) &= (n + 1)N - \sum_{j=0}^{N} ht(q_j) \\ &= (n + 1)N - ht(q_0 \vee \cdots \vee q_N) = (n + 1)N - ht(\mathrm{triv}) = nN + 1.\end{aligned}$$

(the relation $\sum_{j=0}^{k} ht(q_j) = ht(q_0 \vee \cdots \vee q_k)$ is proved by induction on k, using $q_k = (q_0 \vee \cdots \vee q_{k-1})\backslash(q_0 \vee \cdots \vee q_k)$ and $ht(u) + ht(u\backslash v) = ht(v)$). Consider the disjoint union of all the parts of all the terms of q. Among these $nN+1$ parts is the image of the long part of p_i.

The lemma will be proved if we establish that, choosing any of $nN+1$ parts (*i.e.*, choosing a given i in $\{0,\dots,N\}$ and choosing a part of q_i) one may uniquely reconstruct $p \in E^{-1}(q)$ such that p_i is long, with long part sent by E to the chosen part.

Suppose we have chosen q_i and a part a of q_i. Let $\zeta \in a$. We want p_i to have a long part containing the e points in $E^{-1}(\zeta)$. This characterises a unique element of $p_i \in E^{-1}(q_i)$ (Lemma 8.3 (iii)). We now have to see that the remaining p_j's are then uniquely determined. Since $u_i := p_1 \vee \cdots \vee p_i$ must satisfy $E(u_i) = q_1 \vee \cdots \vee q_i$ and $p_i \preccurlyeq u_i$, it is uniquely determined by Lemma 8.3 (iii). Among the constraints is that $p_i = (p_1 \vee \cdots \vee p_{i-1})\backslash u_i$. It is not difficult to see that, in $NCP(e,1,n)$, for all u, the map $[\mathrm{disc}, u] \to [\mathrm{disc}, u], x \mapsto x\backslash v$ is a bijection (this follows from the similar statement in $NCP(1,1,en)$, which itself easily reduces to the case when $u = \mathrm{triv}$, which itself follows from the fact that the operation $x \mapsto \overline{\overline{x}}$ is bijective, which is contained in Lemma 1.8). Applying this to u_i, we find the existence of $x \in NCP(e,1,n)$, uniquely determined, such that $p_1 \vee \cdots \vee p_{i-1} = x\backslash u_i$. Since p_i is long, so is u_i, thus $p_1 \vee \cdots \vee p_{i-1}$ is short. All $p_1,\dots,p_{i-1}$ must be finer than this short element. By Lemma 8.3 (ii), the chain $\int(q_0,\dots,q_{i-1})$ lifts via E to a unique chain in the interval $[0, x\backslash u_i]$. This implies that one may uniquely reconstruct $p_0,\dots,p_{i-1}$.

To reconstruct $p_{i+1},\dots,p_N$, similarly observe that the problem amounts to finding the pre-image of the chain $(q_1 \vee \cdots \vee q_i, q_1 \vee \cdots \vee q_{i+1}, \dots, q_1 \vee \cdots \vee q_{N+1})$, with the condition that the initial term is lifted to u_i. Since u_i is long, this problem admits a unique solution (Lemma 8.3 (iii)). □

The $NCP(e,1,n)$ case of the theorem follows from the above lemma (rephrased to deal with chain rather than derived sequences) and the relation $Z(e,1,n) = (1+n(X-1))Z(1,1,n)$.

We now deal with the $NCP(e,e,n+1)$ case. There are two sorts of chains in $NCP(e,e,n+1)$: *symmetric chains* (chains consisting only of symmetric partitions) and *asymmetric chains* (chains containing at least an asymmetric partition). Via $*$, symmetric N-chains in $NCP(e,e,n+1)$ are in 1-to-1 correspondence with N-chains in $NCP(e,1,n)$. Expanding the relation $Z(e,e,n+1) = (1+\frac{en}{n+1}(X-1))Z(e,1,n)$ and using the $NCP(e,1,n)$ case already proved, we are left with having to prove that the number of asymmetric N-chains is $\frac{en}{n+1}NZ(e,1,n) = \frac{en}{n+1}NZ(2,1,n)$. Using the $NCP(2,2,n+1)$ case already proved by Athanasiadis-Reiner, [AR], the claimed result follows from:

Lemma 8.7. *The number of asymmetric N-chains in $NCP(e,e,n+1)$ is $e/2$ times the number of asymmetric N-chains in $NCP(2,2,n+1)$.*

Proof. Let u be an asymmetric partition, let ζ be the successor of 0 for the counter-clockwise cyclic ordering of the asymmetric part. We say that ζ is the *direction of* u. The *direction* of an asymmetric chain is the direction of the first asymmetric partitions appearing in the chain. Clearly, there are ne possible directions for asymmetric N-chains, and these chains are equidistributed according to the possible directions. To prove the lemma, it is enough to prove that the number of asymmetric N-chains in $NCP(e,e,n+1)$ with direction 1 depends only on n and N, and not on e.

To prove this, observe that an asymmetric chain with direction 1 is uniquely determined by its restriction to $\{0, \zeta_{en}^{-e+1}, \zeta_{en}^{-e+2}, \dots, 1, \dots, \zeta_{en}^{e}\}$ and use the map "$E^{-1} \circ E$" from $NCP(e,e,n+1)$ to $NCP(2,2,n+1)$. □

9. A remark on Broué-Malle-Rouquier generators

In this section, we will show that the BMR presentation defines a monoid which does not embed in the group defined by the same presentation, and that indeed, the submonoid of the braid group of $G(e, e, n+1)$ generated by the BMR generators is not finitely presented.

Let $\mathcal{T}$ denote the BMR generators, and $\mathcal{R}$ the BMR relations. For ease on the eye, we will write 1 for τ_2' ($\leftrightarrow a_n$), 2 for τ_2 ($\leftrightarrow a_0$) and i for τ_i ($\leftrightarrow a_{i-3,i-2}$) for $i = 3, \ldots, n-1$. We now deduce some relations in $B(e, e, n+1)$. Firstly

$$\begin{aligned} 2\ 13213\langle 21\rangle^{e-2} &= 321321\langle 21\rangle^{e-2} = 3213\langle 21\rangle^{e} = 3213\langle 12\rangle^{e} \\ &= 3213\ 1\langle 21\rangle^{e-1} = 32313\langle 21\rangle^{e-1} = 2\ 3213\langle 21\rangle^{e-1}, \end{aligned}$$

so by left cancellation, $13213\langle 21\rangle^{e-2} = 3213\langle 21\rangle^{e-1}$. This new relation can be written $1w = wx$ where w is $3213\langle 21\rangle^{e-2}$ and x is 2 if e is even, and 1 if e is odd. For the same letter x, $\langle 21\rangle^{e}x \equiv \langle 21\rangle^{e+1} \equiv 2\langle 12\rangle^{e}$ hence for any k,

$$21^k 3213\langle 21\rangle^{e-2} \equiv 21^k w = 21wx^{k-1} = 3213\langle 21\rangle^{e}x^{k-1} = 32132^{k-1}\langle 21\rangle^{e} \equiv 32132^k 1\langle 21\rangle^{e-2},$$

so by right cancellation this time, $21^k 3213 = 32132^k 1$.

Let $M_{\mathcal{T}}$ denote the submonoid of the braid group of $G(e, e, n+1)$ generated by the BMR generators. Then $M_{\mathcal{T}}$ is a submonoid of $M = M(e, e, n+1)$, the monoid defined by the presentation given at the beginning of this section. We have a solution to the word problem for M and can calculate least common multiples, and from this we will be able to deduce that there can be no finite presentation for $M_{\mathcal{T}}$. Suppose there were such a finite presentation; then there would be an l for which all relations – which must be words over $\mathcal{T}$ – are of length at most l. For all k, we have $21^k 3213 = 32132^k 1$, so a relation must be applicable to the word $32132^k 1$. Since k can be arbitrarily large, this means that a relation must be applicable to either 32132^k or to $2^k 1$. That is, there would have to exist some (different) word W over $\mathcal{T}$ such that $32132^k = W$ or $2^k 1 = W$. We will see that this cannot be the case.

There is a surjection $B(e, e, n+1) \to B_n$, the braid group of type A on n strings, given by $1, 2 \mapsto \sigma_1$ and $i \mapsto \sigma_{i-1}$ for $i \geq 3$. Thus $2^k 1 \overset{\varphi}{\mapsto} \sigma_1^{k+1}$; so $2^k 1$ can only be rewritten in terms of $\{1, 2\}$. Suppose that $2^k 1 = w2$ for some word w over $\{2, 1\}$; then $2^k 1 \equiv 2^{k-1}21 = 2^{k-1}a_{-n}2$; so by right cancellation, $w = 2^{k-1}a_{-n}$. However $2^{k-1}a_{-n} \equiv a_0^{k-1}a_{-n}$ is in a singleton equivalence class in M, so can never be rewritten in terms of $\{1, 2\}$. Thus it remains to show that 32132^k cannot be rewritten in terms of $\mathcal{T}$.

Let U denote the word $3213\,2^k$. It suffices to show that we cannot rewrite U in terms of $\mathcal{T}_3 := \{1, 2, 3\}$. Since M is Garside, it has the 'reduction property', and so we can quickly determine divisibility in M by a generator using the method of a-chains (see [C2]).

Observe that $U = u\,3213$ where $u = a_{-n}^k$; u is not divisible by 1, 2, or 3, so is certainly not rewriteable over these letters. Also, neither 1 nor 2 right divides $u\,3$ so U cannot be $\mathcal{T}_3$-rewritten to end with 213. Since $32 = a_1 3$, we have $U = v\,313$ where $v = ua_1$, which is in a singleton equivalence class in M, so not $\mathcal{T}_3$-rewriteable; so U cannot be $\mathcal{T}_3$-rewritten to end with 313. Furthermore, $v\,3$ is not right divisible by 1, so U cannot be $\mathcal{T}_3$-rewritten to end with 113. Thus U cannot be $\mathcal{T}_3$-rewritten to end with 13.

Next, write $U = w\,323$ where $w = ua_{1-n}$; w is not $\mathcal{T}_3$-rewriteable; and $w\,3$ is not right divisible by 1 or 2. Thus U cannot be rewritten over $\{1, 2, 3\}$ to end with 23. Finally, $U = x\,3^2$ where

$x = ua_{1-n}a_1$; x is not right divisible by 1,2 or 3, so we have that U cannot be rewritten to end with 33 either. This exhausts all the possibilites for $\mathcal{T}_3$-rewriting U to end with 3.

Now write $U = y\,321$ where $y = ua_{1,n}$. This is not right divisible by $1, 2$ or 3; so not $\mathcal{T}_3$-rewriteable over these letters. Moreover, $y\,3$ is not right divisible by 1 or 2; and $y\,32$ is not right divisible by 1. Thus we have that U cannot be rewritten to end with 12 or 11. Write $U = v\,131$ where $v = ua_1$, which is not right divisible by $1, 2$ or 3; furthermore, $v1$ is not divisible by 2 or 3. In this way we have exhausted all the possibilites for rewriting U over $\{1, 2, 3\}$ ending with 1.

Finally, we want to show that U cannot be rewritten to end with 2. By right cancellation, assume that $k = 0$. Now $3213 = a_{1-n}232$, but $a_{1-n}23$ is not left divisible by $1, 2$ or 3, so cannot be rewritten over $\{1, 2, 3\}$. Thus

Proposition 9.1. *There is no finite presentation for the submonoid of the braid group of type $G(e, e, r)$ generated by the Broué-Malle-Rouquier generators.* □

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